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ON LINEAR WAVE MOTIONS IN MAGNETIC-VELOCITY SHEARS

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CONTENTS

The propagation properties of linear wave motions in magnetic and/or velocity shears which vary in one coordinate z (say) are usually governed by a second order linear ordinary differential equation in the independent variable z. It is proved that associated with any such differential equation there always exists a quantity $\mathscr{A}$ which is independent of z. By employing $\mathscr{A}$ as a measure of the intensity of the wave, this result is used to investigate the general propagation properties of hydromagnetic-gravity waves (e.g. critical level absorption, valve effects and wave amplification) in magnetic and/or velocity shears, using a full wave treatment. When variations in the basic state are included, the governing differential equation usually has more singularities than it has in the W.K.B.J. approximation, which neglects all variations in the background state. The study of a wide variety of models shows that critical level behaviour occurs only at the singularities predicted by the W.K.B.J. approximation. Although the solutions of the differential equation are necessarily singular at the irregularities whose presence is solely due to the inclusion of variations in the basic state, the intensity of the wave (as measured by $\mathscr{A}$) is continuous there. Also the valve effect is found to persist whatever the relation between the wavelength of the wave and the scale of variations of the background state. In addition, it is shown that a hydromagnetic-gravity wave incident upon a finite magnetic and/or velocity shear can be amplified (or over-reflected) in the absence of any critical levels within the shear layer. In a Boussinesq fluid rotating uniformly about the vertical, wave amplification can occur if the horizontal vertically sheared flow and magnetic field are perpendicular. In a compressible isothermal fluid, on the other hand, wave amplification not only occurs in both magnetic-velocity and velocity shears but also in a magnetic shear acting alone.

1. Introduction

The propagation properties of linear wave motions in magnetic and non-magnetic systems, with applications varying from atmospheres to interiors of the planets, have been studied by many authors (see, for example, Lighthill 1960; Braginsky 1964; Hide 1966; Bretherton 1966; Booker & Bretherton 1967; Lighthill 1967; Jones 1968; McKenzie 1972; Eltayeb 1972; Acheson & Hide 1973). Bretherton (1966) studied the propagation of internal gravity waves in a shear flow using the W.K.B.J. approximation (which neglects variations in the basic state) and showed that at a height $z = z_c$ (say), where the horizontal phase speed of the wave matches the Doppler-shifted frequency, a 'critical level' exists and the wave is neither reflected nor transmitted there but is absorbed. In a more rigorous treatment Booker & Bretherton (1967) re-examined the same problem and showed that the waves are transmitted across the critical level but are heavily attenuated when the Richardson number (which is a dimensionless parameter used as a measure of the strength of the shear) is of order unity or larger. The reflexion and transmission of gravity waves by a finite shear has recently been studied by Eltayeb & McKenzie (1975) for all positive Richardson numbers. They found that the waves give away some of their energy to the basic flow at the critical level for all positive values of the Richardson number and also the reflected wave is amplified for small values of the Richardson number provided a critical level exists within the layer. These results, as well as those of Eltayeb & McKenzie (1977), are consistent with the conjecture, proposed by McKenzie (1972), that wave amplification (or over-reflexion) is due to the interaction of negative and positive energy waves. Since all those studies have been carried out for an incompressible Boussinesq fluid, it would be interesting to know whether the conjecture will survive in a compressible medium.

Acheson (1972, 1973) studied the propagation of hydromagnetic waves in rotating systems and in non-rotating systems in the presence of gravity, using the W.K.B.J. approximation, and showed that in certain circumstances hydromagnetic waves can exhibit 'valve' effects, i.e. they can penetrate a critical level from one side only. The question to be asked here is: does the valve effect persist in a full wave treatment which includes variations in the basic state?

Bretherton & Garrett (1968) (see also Hayes 1970) examined the propagation of gravity waves and defined a quantity they christened, 'wave action flux' which is independent of the coordinate in which the shear varies. This quantity, which is continuous everywhere except at the critical levels, supplies a convenient measure for the intensity of gravity waves and hence can be used to examine the effect of the critical level on the waves. Jones (1967) studied the propagation of internal gravity waves in the presence of a uniform vertical angular velocity and showed that the vertical transport of angular momentum is conserved except at the singularities predicted by the W.K.B.J. approximation. Rudraiah & Venkatachalappa (1972) extended Jones's analysis to include the effects of a uniform transverse magnetic field to show that the vertical transport of angular momentum is conserved. Grimshaw (1975) studied the propagation of gravity waves in the presence of a uniform angular velocity which is inclined to the vertical in a vertically sheared flow and constructed a quantity which is independent of the vertical coordinate. However, no simple physical interpretation of the quantity was found. It may then be asked: Can all linear wave motions in a basic state which depends on one coordinate possess a quantity which is independent of that coordinate?

The equations governing linear wave motions in one dimensional magnetic and/or velocity shears can always be reduced to a second order linear ordinary differential equation (see, for example, Booker & Bretherton 1967; Jones 1967; Eltayeb & McKenzie 1977). A common feature of the equations obtained for various models is that when variations in the basic state are retained (as should be the case in a full wave treatment), the equation has more singularities than when these variations are neglected (as in the case of the W.K.B.J. approximation). The question then arises: do the singularities introduced by the inclusion of variations in the basic state behave like critical levels or are they singularities of a different nature, and if so, what are their effects on the waves?

This work is concerned with the above questions.

In §2 we prove that *associated with any second order linear ordinary differential equation in an independent variable z there always exists a quantity* $\mathscr{A}$ *which is independent of z*. This result is used to study the propagation properties of hydromagnetic gravity waves in magnetic-velocity shears in the presence of various constraints. Three particularly different models are studied in §§3–5. The propagation properties of hydromagnetic-inertial-gravity waves in a Boussinesq incompressible fluid are examined in §3. In §4 we study magnetic-acoustic-gravity waves and in §5 an anisotropic system in which gravity and magnetic field are both inclined to the direction of the magnetic shear is investigated. §6 is devoted to a few concluding remarks.

2. The existence of a wave-invariant

The linear treatment of the propagation of wave motions in magnetic and/or velocity shears, which vary in one coordinate z (say), in compressible and incompressible fluids, usually leads to an equation of the type

$$a(z)W''(z)+2b(z)\,W'(z)+c_1(z)\,W(z) = 0, \tag{2.1}$$

where a, b, c_1 are functions of the basic state variables and W is the component of velocity in the direction of the shear, the accent denoting differentiation with respect to the argument (see, for example, Booker & Bretherton 1967; Jones 1967; Eltayeb & McKenzie 1977, and §§3–5 below). Suppose the shear layer is bounded by $z = z_1$ and $z = z_2$ and assume that W is restricted by the boundary conditions

$$W'(z_1) = \alpha_1 W(z_1), \quad W'(z_2) = \alpha_2 W(z_2), \tag{2.2}$$

in which α_1 and α_2 are prescribed constants.

The transformation

$$W = h\psi, \quad h = \exp -\int (b/a)\,\mathrm{d}z, \tag{2.3}$$

reduces (2.1) to

$$\psi''+C\psi = 0, \tag{2.4}$$

where

$$C = [a(c_1-b')+ba'-b^2]/a^2. \tag{2.5}$$

To construct an invariant (with respect to z) of (2.1) and (2.2), we need to consider three different cases.

Case (*i*). If C is *real*, we take the complex conjugate of (2.4), multiply it by ψ and subtract the result from (2.4) after it has been multiplied by the complex conjugate of ψ to get

$$\bar{\psi}\psi'' - \psi\bar{\psi}'' = 0, \tag{2.6}$$

where the 'bar' denotes the complex conjugate. Equation (2.6) and the boundary conditions (2.2) immediately show that the quantity

$$\mathscr{A} = \mathrm{Re}\,(-\mathrm{i}\psi'\bar{\psi}), \tag{2.7}$$

is independent of z.

Case (*ii*). For *imaginary* values of C we find, in a similar way, that

$$\mathscr{A} = \mathrm{Re}\,(\psi'\bar{\psi}), \tag{2.8}$$

is independent of z.

Case (*iii*). In the case of a *complex* C $(= C_r + \mathrm{i}C_i)$, we set

$$\psi = \phi + \mathrm{i}F, \tag{2.9}$$

and substitute in (2.1) to get $$\phi'' + C_r\phi = -(F'' + C_r F + C_i\psi). \tag{2.10}$$

We now choose ϕ such that $$\phi'' + C_r\phi = 0, \tag{2.11}$$

and then F is a *particular* integral of $$F'' + C_r F = -C_i\psi. \tag{2.12}$$

Since C_r is real then (2.11) leads to the quantity

$$\mathscr{A} = \mathrm{Re}\,(-\mathrm{i}\phi'\bar{\phi}) \tag{2.13}$$

being independent of z, as in case (i) above. Now ϕ is defined by (2.9) and $\mathscr{A}$ can be written as

$$\mathscr{A} = \mathrm{Re}\,[-\mathrm{i}(\psi' - \mathrm{i}F')\,(\bar{\psi} + \mathrm{i}\bar{F})]. \tag{2.14}$$

Written in this form $\mathscr{A}$ is determined by solving (2.4) together with the boundary conditions to determine the (unique) solution ψ and then F is obtained as a particular integral of (2.12). Thus, unless there are some inherent pathological defects in the original problem leading to non-uniqueness in ψ, the quantity $\mathscr{A}$ will be unique.

It should be pointed out here that the above proof breaks down at the singular points of (2.4), i.e. at points z, where $|C| = \infty$. It then follows that *with the well-posed system* (2.1) *and* (2.2) *there always exists a unique quantity* $\mathscr{A}$ *which is invariant with respect to z except possibly at the singular points of* (2.4).

As a corollary of this we observe that if a singular point, z_0, of (2.1) is *not* a singular point of (2.4) then $\mathscr{A}$ is continuous across z_0 (see §§3 and 4 below).

Returning to linear wave motions in magnetic and velocity shears which are governed by (2.1) and (2.2), we see that these motions will always possess a 'wave-invariant'. The term wave-invariant, which will be used throughout this paper to refer to $\mathscr{A}$, is suitable since the parameters of the system, such as the frequency of the wave and the wavenumber in the direction perpendicular to z, are kept fixed. In these situations $\mathscr{A}$ for gravity waves is proportional to the flux of wave action of gravity waves (Bretherton & Garrett 1968; Hayes 1970).

If the finite shear is bounded by two uniform basic states in the spaces $z < z_1$, and $z > z_2$ and provided there are no singularities at $z = z_1$ and $z = z_2$ then the wave-invariant is continuous everywhere except possibly at the singular points within the shear layer. This result will be exploited in the next three sections to examine the natures of the critical levels that can arise in magnetic and/or velocity shears, as well as to study the phenomenon of wave amplification (or over-reflexion) by a finite shear using a *full* wave treatment.

3. Hydromagnetic-gravity waves in a rotating Boussinesq fluid

3.1. *Formulation*

Consider an inviscid Boussinesq fluid of infinite electrical conductivity rotating uniformly with angular velocity $\boldsymbol{\Omega}$. Take a cartesian system of coordinates (x, y, z), rotating with the fluid, such that the z-axis is vertically upwards and parallel to $\boldsymbol{\Omega}$ and the x and y-axes any two perpendicular horizontal directions. The equations of motion, continuity, induction and Gauss's law admit a basic state in which

$$\boldsymbol{u}_0 = U(z)\,\hat{\boldsymbol{x}}, \quad \boldsymbol{B}_0 = B(z)\,\hat{\boldsymbol{y}}, \tag{3.1}$$

where $\boldsymbol{u}$ and $\boldsymbol{B}$ are the velocity and magnetic induction respectively, provided the fluid pressure p_0 satisfies the equation

$$2\rho_0 \boldsymbol{\Omega} \wedge \boldsymbol{u}_0 = -\nabla(p_0 + \boldsymbol{B}_0^2/2\mu) + \rho_0\,\boldsymbol{g}, \tag{3.2}$$

in which ρ_0 is the density, μ the magnetic permeability and $\boldsymbol{g}$ the gravitational acceleration (assumed constant). This situation may be realized in the atmosphere if both the horizontal component of the angular velocity and the variations of the vertical component of the angular velocity of the earth are neglected, assumptions which may be justifiable in the case of small period disturbances (of the order less than a day) as opposed to planetary waves for which the β-plane approximation provides a reasonable approximation (Longuet-Higgins 1965). Hydromagnetic-gravity waves in rotating fluids are also relevant to the Earth's and planetary interiors (Hide 1966; Braginsky 1967).

If we assume an equation of state

$$\frac{\partial}{\partial z}\ln \rho_0 = -\beta \equiv \text{constant} \tag{3.3}$$

then (3.2) and (3.3) give the thermal wind equation

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial y} = \frac{2\Omega}{g}(U' - \beta U), \tag{3.4}$$

where the prime, as always in this paper, denotes differentiation with respect to the argument. In the Boussinesq approximation, which is adopted here, the last term on the right hand side of (3.4) is negligible.

We now assume small disturbances $\boldsymbol{u}$, $\boldsymbol{b}$ and ρ_1 in velocity, magnetic induction and density respectively and divide them into normal modes

$$F(x, y, z, t) = \operatorname{Re}\{F(z)\exp\mathrm{i}(wt - kx - ly)\}, \tag{3.5}$$

in which Re denotes 'the real part of', to find that the vertical component of $\boldsymbol{u}$ satisfies equation (2.1) with

$$a = \hat{\omega}^2 - l^2V^2 - 4\Omega^2\hat{\omega}^2/(\hat{\omega}^2 - l^2V^2),$$

$$b = -U'\left[-2\Omega l\mathrm{i} + \frac{klV^2}{\hat{\omega}} + \frac{4\Omega^2\hat{\omega}^3 k}{(\hat{\omega}^2 - l^2V^2)}\right] - l^2VV'\left[1 + \frac{4\Omega^2\hat{\omega}^2}{(\hat{\omega}^2 - l^2V^2)}\right],$$

$$c_1 = U''[k(\hat{\omega}^2 - l^2V^2)/\hat{\omega} + 2\Omega l\mathrm{i}] + 2k^2U'^2\left[\frac{2\Omega l\mathrm{i}}{k\hat{\omega}} - \frac{l^2V^2}{\hat{\omega}^2}\right] - \frac{2l^2VV'kU'}{\hat{\omega}^2} - (\hat{\omega}^2 - l^2V^2 - N^2)(k^2+l^2), \tag{3.6}$$

where V is the Alfvén speed, $\hat{\omega}$ the Doppler-shifted frequency and N is the Brunt-Väisällä frequency

$$V^2 = B^2/\mu\rho_0, \quad \hat{\omega} = \omega - kU, \quad N^2 = \beta g. \tag{3.7}$$

The x and y components of velocity, total perturbation pressure Π and magnetic induction (b_x, b_y, b_z) are given by

$$u = \frac{\mathrm{i}l^2(\hat{\omega}^2 - l^2V^2)U'W - \hat{\omega}\{\mathrm{i}k(\hat{\omega}^2 - l^2V^2) + 2\Omega\hat{\omega}l\}W'}{(k^2+l^2)\hat{\omega}(\hat{\omega}^2 - l^2V^2)},$$

$$v = \frac{-\mathrm{i}kl(\hat{\omega}^2 - l^2V^2)U'W - \hat{\omega}\{\mathrm{i}l(\hat{\omega}^2 - l^2V^2) - 2\Omega\hat{\omega}k\}W'}{(k^2+l^2)\hat{\omega}(\hat{\omega}^2 - l^2V^2)},$$

$$\Pi = \frac{\{k(\hat{\omega}^2 - l^2V^2) + 2\hat{\omega}l\mathrm{i}\}U'W + \hat{\omega}aW'}{\mathrm{i}\hat{\omega}^2(k^2+l^2)},$$

$$b_x = -\frac{lB}{\hat{\omega}}u + \frac{\mathrm{i}lB}{\hat{\omega}^2}W, \quad b_y = -\frac{lB}{\hat{\omega}}v + \frac{\mathrm{i}B'}{\hat{\omega}}W, \quad b_z = -\frac{lB}{\hat{\omega}}W. \tag{3.8}$$

Employment of the transformation (2.3) reduces the equation for W to (2.4) for ψ provided that

$$\begin{aligned} a^2C = {} & a(k\hat{\omega}U'' + l^2VV'')[1 + 4\Omega^2\hat{\omega}^2/(\hat{\omega}^2 - l^2V^2)^2] \\ & + k^2U'^2\left[\frac{4\Omega^2l^2}{k^2} + l^2V^2\left\{1 + \frac{12\Omega^2\hat{\omega}^2}{(\hat{\omega}^2 - l^2V^2)^2} - \frac{4\Omega^2}{(\hat{\omega}^2 - l^2V^2)}\right\}\right. \\ & \left. + \frac{4\Omega^2\hat{\omega}^4}{(\hat{\omega}^2 - l^2V^2)^2}\left\{3 + \frac{4\Omega^2}{(\hat{\omega}^2 - l^2V^2)} - \frac{12\Omega^2\hat{\omega}^2}{(\hat{\omega}^2 - l^2V^2)^2}\right\}\right] \\ & + 2l^2VV'kU'\left\{1 + \frac{4\Omega^2\hat{\omega}^2}{(\hat{\omega}^2 - l^2V^2)^2} + \frac{4\Omega^2}{(\hat{\omega}^2 - l^2V^2)^2}\left[3\hat{\omega}^2 + 2l^2V^2 - \frac{12\Omega^2\hat{\omega}^4}{(\hat{\omega}^2 - l^2V^2)^2}\right]\right\} \\ & + \hat{\omega}l^2V'^2\left\{\left[1 + \frac{4\Omega^2\hat{\omega}^2}{(\hat{\omega}^2 - l^2V^2)^2}\right]\left[1 - \frac{4\Omega^2}{(\hat{\omega}^2 - l^2V^2)}\right] + \frac{4\Omega^2l^2V^2}{(\hat{\omega}^2 - l^2V^2)^2}\left[5 - \frac{12\Omega^2\hat{\omega}^2}{(\hat{\omega}^2 - l^2V^2)^2}\right]\right\} \\ & - (k^2+l^2)(\hat{\omega}^2 - l^2V^2 - N^2)a. \end{aligned} \tag{3.9}$$

Since C is real, the wave-invariant is given by (2.7) and (2.4). Although the level $\hat{\omega} = 0$ is singular in the equation for W, the expression (3.9) for C shows that ψ is regular there. Consequently the wave-invariant is continuous across such levels.

In the absence of the magnetic field Jones (1967) has shown that the vertical flux of angular momentum is independent of z. By solving (2.1) and (3.6) with $V = 0$ he concluded that the vertical flux of angular momentum is discontinuous across the levels $a = 0$ (i.e. $\hat{\omega} = \pm 2\Omega$) but is *continuous* across $\hat{\omega} = 0$. The case of a uniform magnetic field (i.e. V uniform but non-zero) was studied by Rudraiah & Venkatachalappa (1972). They calculated the vertical flux of angular momentum taking into account the effects of the uniform magnetic field to show that the vertical flux of angular momentum is independent of z except at the singularities of (2.1). By computing the solutions near these levels they concluded that the vertical flux of angular momentum was discontinuous across all singularities including $\hat{\omega} = 0$. In search of a physical interpretation of the wave-invariant derived above we have calculated $\mathscr{A}$ as given by (2.7) and found that

$$\mathscr{A} = \frac{\hat{\omega}^2}{a|h|^2}\mathrm{Re}\,[\mathrm{i}a\overline{W}W' - 2\Omega lU'|W|^2], \tag{3.10}$$

from which we can see that the quantity within the square brackets is that defined as G in both Jones (1967) and Rudraiah & Venkatachalappa (1972). Now in a straightforward manner it can be shown, using the expressions (3.6) for a and b, that

$$\hat{\omega}^2/a|h|^2 = 1. \tag{3.11}$$

This shows that the wave-invariant is proportional to the vertical flux of angular momentum. Consequently the vertical flux of angular momentum is *continuous* across $\hat{\omega} = 0$ whether the magnetic field is uniform (and whether it vanishes or not) or non-uniform.

3.2 *The wave normal surfaces*

In the W.K.B.J. approximation, i.e. when variations in the basic state are ignored by setting $U' = V' = 0$, a *local* dispersion relation is obtained

$$k_z^2 = \frac{(k^2+l^2)\,(\hat{\omega}^2-l^2V^2)\,(N^2+l^2V^2-\hat{\omega}^2)}{(\hat{\omega}^2-l^2V^2)^2-4\Omega^2\hat{\omega}^2}. \tag{3.12}$$

In a medium in which U and V are uniform this relation is strictly true.

Since the waves are three dimensional we shall study the cross sections of these surfaces in planes l = constant and planes k = constant. In planes l = constant and in the absence of a flow (i.e. $U = 0$) equation (3.12) can be written in the form

$$k_z^2 = \tau^2(k^2+l^2), \quad \tau^2 = \frac{(\omega^2-l^2V^2)\,(N^2+l^2V^2-\omega^2)}{(\omega^2-l^2V^2)^2-4\Omega^2\omega^2}. \tag{3.13}$$

Thus propagation is possible only if $\tau^2 > 0$. This yields the conditions

$$\begin{gathered} l^2V^2+2\Omega^2-2\Omega(\Omega^2+l^2V^2)^{\frac{1}{2}} < \omega^2 < l^2V^2, \\ \min\,[N^2+l^2V^2,\quad l^2V^2+2\Omega^2+2\Omega(\Omega^2+l^2V^2)^{\frac{1}{2}}] < \omega^2 \\ < \max\,[N^2+l^2V^2,\, l^2V^2+2\Omega^2+2\Omega(\Omega^2+l^2V^2)^{\frac{1}{2}}], \end{gathered} \tag{3.14}$$

which show that the first range of ω^2 is made possible by the presence of the magnetic field (i.e. the slow wave) while the second is that present in the absence of the magnetic field but is here modified by its presence (see figure 1*a*).

When a flow is present, the wave normal curves in the (k, k_z) plane have four asymptotes occurring at values of k given by

$$\left.\begin{aligned} k_{\mathrm{I1}} &= U^{-1}[\omega-\Omega-(\Omega^2+l^2V^2)^{\frac{1}{2}}], \\ k_{\mathrm{A1}} &= U^{-1}[\omega+\Omega-(\Omega^2+l^2V^2)^{\frac{1}{2}}], \\ k_{\mathrm{A2}} &= U^{-1}[\omega-\Omega+(\Omega^2+l^2V^2)^{\frac{1}{2}}], \\ k_{\mathrm{I2}} &= U^{-1}[\omega+\Omega+(\Omega^2+l^2V^2)^{\frac{1}{2}}]. \end{aligned}\right\} \tag{3.15}$$

Also k_z vanishes when k takes the values

$$\left.\begin{aligned} k_{\mathrm{I1}}^{(0)} &= U^{-1}[\omega-(N^2+l^2V^2)^{\frac{1}{2}}], \\ k_{\mathrm{A1}}^{(0)} &= U^{-1}(\omega-lV), \\ k_{\mathrm{A2}}^{(0)} &= U^{-1}(\omega+lV), \\ k_{\mathrm{I2}}^{(0)} &= U^{-1}[\omega+(N^2+l^2V^2)^{\frac{1}{2}}] \end{aligned}\right\} \tag{3.16}$$

The suffices I and A refer to 'inertial' and 'Alfvén' respectively to identify those modes which are present in the absence of the magnetic field and those which vanish when the magnetic field is ineffective. This notation is adopted in figure 1*b*, where the cross sections of the wave normal surfaces in the (k, k_z) plane are sketched. It may be noted that all waves appear to have critical levels and that the waves can propagate on one side of the critical level. However, the cross sections of the wave normal surfaces in the (l, k_z) plane can be different from those in the (k, k_z) plane in that some of the curves do not appear to have critical levels. (See figure 2.) When the wave normal surfaces are viewed three dimensionally, however, four critical levels are possible.

The advantage of using the W.K.B.J. approximation (see, for example, Reid 1965) to derive a local dispersion relation is that a qualitative treatment for general basic states becomes possible.

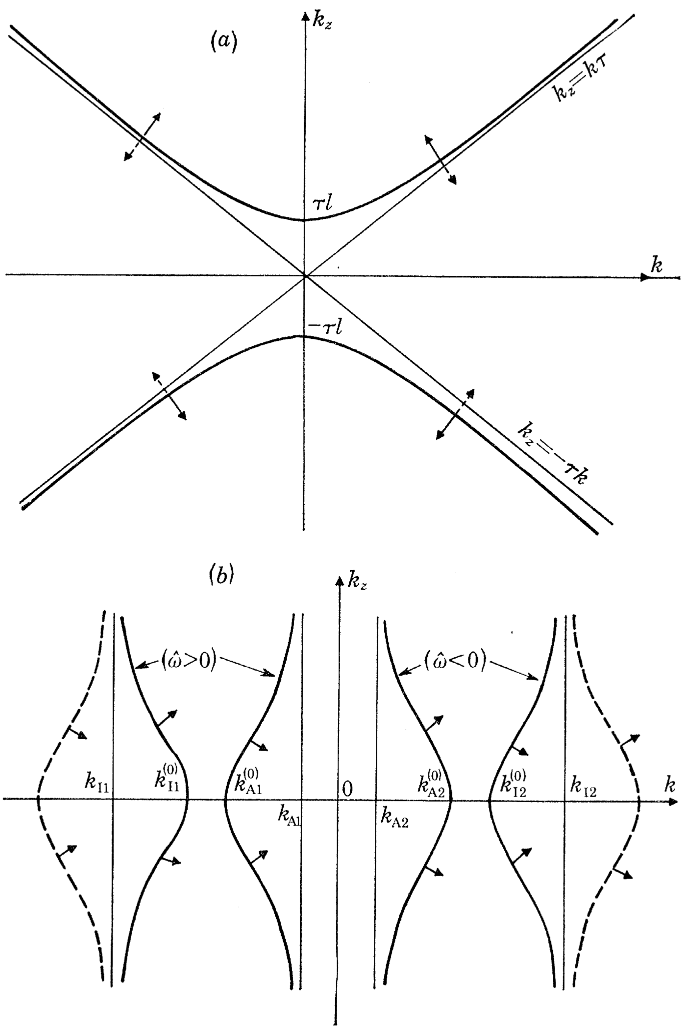

FIGURE 1. The cross section of the wave normal surfaces for hydromagnetic-inertial-gravity waves in the (k, k_z) plane for a fixed non-zero value of l.

(*a*) In the absence of a flow. The continuous arrows represent the direction of the group velocity when $N^2 < 4\Omega^2$ and

$$(4\Omega^2 - N^2)^{-1}\{-l^2V^2N^2 + 2\Omega lV[4\Omega^2 l^2V^2 + N^2(4\Omega^2 - N^2)]^{\frac{1}{2}}\} < \omega^2 < l^2V^2 + 2\Omega^2 + 2\Omega(\Omega^2 + l^2V^2)^{\frac{1}{2}},$$

while the discontinuous arrows apply in the remainder of the ranges specified in equation (3.14). The two straight lines $k_z = \pm\tau k$ are the asymptotes to the curves.

(*b*) In the presence of a flow. The position of the k_z-axis depends on the sign of the k's. Here it is shown for $k_{A1} < 0$. The continuous (discontinuous) curves correspond to $N^2 \lessgtr 2\Omega[\Omega + (\Omega^2 + l^2V^2)^{\frac{1}{2}}]$. For notation see equations (3.15) and (3.16).

In particular the dispersion relation can be utilized to construct the wave normal surfaces and the ray trajectories of waves with a wavelength much smaller than the scale of variations of the basic state. For fixed l the frequency ω and the wavenumber k are conserved along a ray path in the (x, z) plane (see Longuet-Higgins 1965; Lighthill 1967). Similarly, for fixed k the frequency ω and the wavenumber l are conserved along a ray trajectory in the (y, z) plane. Thus from the dispersion relations the local group velocity can be computed and the direction of the ray determined. By repeating the process at different heights the ray trajectories can be constructed (see figure 3).

The cross sections of the wave normal surfaces in planes k = constant and in planes l = constant show that waves can only propagate on one side of the critical level. The implication of this result

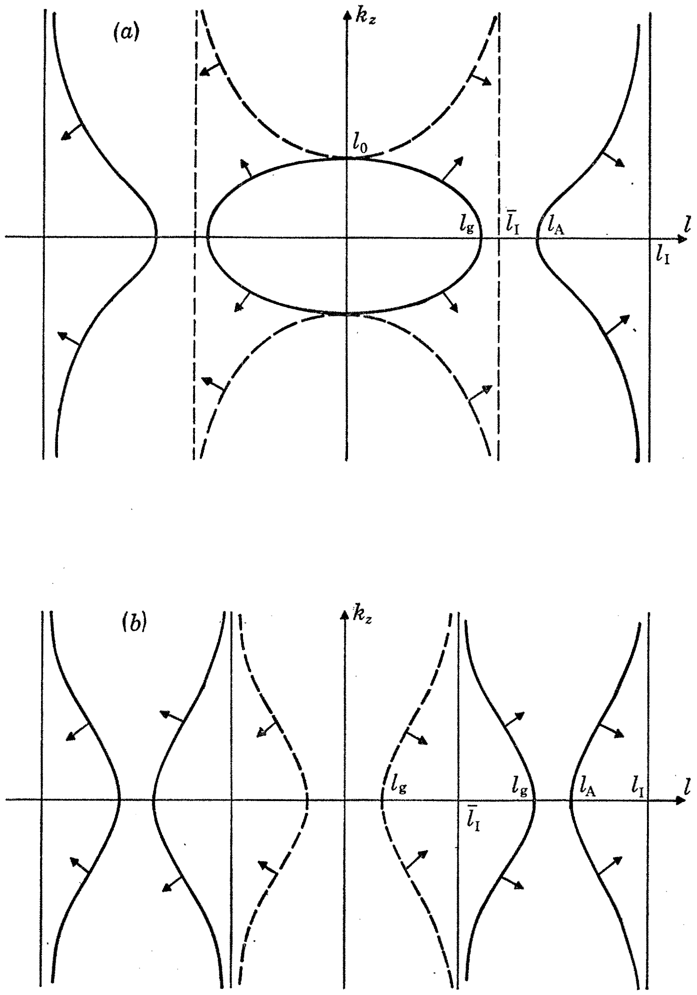

FIGURE 2. The cross sections of the wave normal surfaces for hydromagnetic-inertial-gravity waves in the (l, k_z) plane for fixed k. Here $l_A = \hat{\omega}/V$,

$$l_I = [\hat{\omega}(\hat{\omega}+2\Omega)]^{\frac{1}{2}}/V, \quad \bar{l}_I = [\hat{\omega}(\hat{\omega}-2\Omega)]^{\frac{1}{2}}/V,$$

$$l_g = (\hat{\omega}^2-N^2)^{\frac{1}{2}}/V, \quad l_0 = (\hat{\omega}^2-N^2)^{\frac{1}{2}}(4\Omega^2-\hat{\omega}^2)^{-\frac{1}{2}}.$$

(a) The continuous curves are valid if $N^2 < \hat{\omega}^2 < 4\Omega^2$. If $N^2 > \hat{\omega}^2 > 4\Omega^2$, the curves are composed of the discontinuous ones plus the two branches with the asymptotes.

(b) $\hat{\omega}^2 > N^2, 4\Omega^2$. Note that $l_g \gtrless \bar{l}_I$ if $N^2 \lessgtr 2\Omega\hat{\omega}$. This is indicated by the discontinuous curves near $l = \bar{l}_I$. When $\hat{\omega}^2 < N^2, 4\Omega^2$ the branches near $l = \pm\bar{l}_I$ vanish and the other two remain.

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is that a wave approaching a critical level is *completely* absorbed there. However, near a critical level the local vertical wavelength tends to infinity and hence the assumption on which the W.K.B.J. approximation is based breaks down. Thus a study of the properties of the waves in the vicinity of the critical level must take account of the variations in the basic state. This will be discussed in §3.3 below.

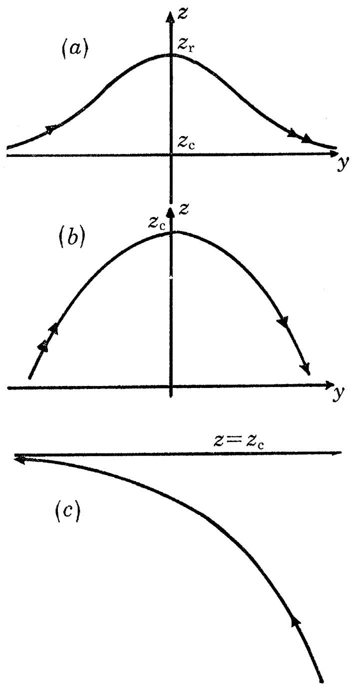

FIGURE 3. The cross sections in the (l, k_z) plane of some of the types of ray trajectories that can arise in magnetic-velocity shears in a rotating incompressible fluid under the influence of gravity.

(*a*) Rays starting near critical levels in magnetic-velocity shears increasing with height. z_c is the critical level and z_r is the reflexion point (see figure 2).

(*b*) Rays corresponding to the closed loop of figure 2*a* in a wind increasing with height.

(*c*) Rays corresponding to the discontinuous curves of figure 2*a*.

3.3 *Critical levels*

Equation (3.9) shows that the equation (2.5) for ψ has two singularities occurring where $a = \hat{\omega}^2 - l^2V^2 = 0$. In this subsection we shall examine the solutions in the neighbourhood of these levels, taking variations in the basic state into account, and determine the effect of these singularities on the intensity of the wave, as measured by the wave-invariant $\mathscr{A}$.

Near the singularity $a = 0$ we have, at leading order,

$$\psi'' + (E^2 + \tfrac{1}{4})\psi/(z-z_c)^2 = 0,$$

$$E = |\Omega l U_c' \hat{\omega}_c / [(\hat{\omega}_c^2 + l^2V_c^2)kU_c' + 2\hat{\omega}_c V_c V_c' l^2]|, \tag{3.17}$$

in which the subscript c means that the quantity is evaluated at the critical level $z = z_c$. The solution of (3.17) has a branch point at $z = z_c$. The matching procedure across $z = z_c$ is straightforward (see Miles 1961; Booker & Bretherton 1967; Baldwin & Roberts 1970). We get

$$\psi = \begin{cases} A(z_c - z)^{\frac{1}{2}+iE} + B(z_c - z)^{\frac{1}{2}-iE} & (z < z_c) \\ A^*(z - z_c)^{\frac{1}{2}+iE} + B^*(z - z_c)^{\frac{1}{2}-iE} & (z > z_c) \end{cases}, \tag{3.18}$$

where $$A^* = \mp iA\exp\{\pm \pi E\}, \quad B^* = \mp iB\exp\{\mp \pi E\}, \tag{3.19a}$$

in which the upper (lower) sign applies when

$$\zeta = (\hat{\omega}_c^2 + l^2 V_c^2) k U_c' + 2\hat{\omega}_c l^2 V_c V_c' \gtrless 0. \tag{3.19b}$$

The evaluation of the wave-invariant below and above the critical level yields

$$\left.\begin{aligned} \mathscr{A}_{\text{below}} &= E(|A|^2 - |B|^2), \\ \mathscr{A}_{\text{above}} &= E(|A|^2 \exp\{\pm 2\pi E\} - |B|^2 \exp\{\mp 2\pi E\}). \end{aligned}\right\} \tag{3.20}$$

For the sake of convenience we shall refer to the solutions of amplitudes A and B by the 'A solution' and the 'B solution' respectively. Now

$$|z - z_c|^{\frac{1}{2} \pm iE} = |z - z_c|^{\frac{1}{2}} \exp\{\pm iE \ln|z_c - z|\}.$$

Thus A and B solutions have *local* vertical wavenumbers, respectively,

$$k_z = \mp E/(z_c - z), \tag{3.21}$$

which shows that on either side of the critical level, the A and B solutions represent ascending and descending waves. To determine which is the ascending wave we need to know which of the four critical levels of (3.12) we are dealing with in order to determine whether the ascending wave is associated with a positive or a negative k_z. For example, in the case of the k_{I1} (< 0) critical level of figure 1*b* (a situation which may occur in a wind increasing with height in the presence of a magnetic field), and provided $N^2 < 2\Omega[\Omega + (\Omega^2 + l^2 V^2)^{\frac{1}{2}}]$, the ascending wave is associated with $k_z > 0$. Thus the B solution represents an ascending wave and then the A solution must refer to a descending wave. On the other hand, for a critical level k_{A1} (> 0) of figure 1*b*, in the same circumstances, the A solution is an ascending wave while the B solution is a descending wave. Also in both cases the amplitudes of the waves are attenuated by a factor $\exp(-\pi E)$ as they cross to the other side of the critical level. Thus although the waves are heavily attenuated in crossing the singularity especially for high rotation rates (i.e. large E), nevertheless they still propagate on the other side of the critical level. Similar arguments apply to the other critical levels.

We now investigate the singularity $\hat{\omega}^2 = l^2 V^2$, which incidentally is a reflexion point in the W.K.B.J. treatment as is clearly shown by equation (3.12). Here it is found that

$$\psi'' - \tfrac{3}{4}\psi/(z - z_c)^2 = 0. \tag{3.22}$$

The legitimate solution can be shown to be

$$\psi = \left\{\begin{matrix} A_1(z_c - z)^{-\frac{1}{2}} + B_1(z_c - z)^{\frac{3}{2}} & (z < z_c), \\ \mp iA_1(z - z_c)^{-\frac{1}{2}} \mp iB_1(z - z_c)^{\frac{3}{2}} & (z > z_c), \end{matrix}\right\} \tag{3.23}$$

where the upper (lower) sign is taken according to

$$\hat{\omega}_c(\hat{\omega}_c k U_c' + l^2 V_c V_c') \gtrless 0.$$

When the wave-invariant is calculated on both sides of the singularity, it is found that

$$\mathscr{A}_{\text{below}} = \mathscr{A}_{\text{above}} = 2\,\text{Re}\,(i\bar{A}_1 B_1). \tag{3.24}$$

The wave-invariant then is *continuous* across this singularity, which is absent in the W.K.B.J. approximation, although the solution ψ is singular there.

3.4. *Reflexion by and transmission through a finite shear*

Suppose that the basic state (3.1) is such that

$$U, B = \begin{cases} 0, B_1 & \text{for} \quad z \leqslant 0 \quad \text{(region I)}, \\ U(z), B(z) & \text{for} \quad 0 \leqslant z \leqslant L \quad \text{(region II)}, \\ U_3, B_3 & \text{for} \quad z \geqslant L \quad \text{(region III)}, \end{cases} \tag{3.25}$$

and assume that U and B (or equivalently V) and their derivatives are continuous across $z = 0, L$. The continuity of W and Π across $z = 0, L$ then yields (Eltayeb & McKenzie 1977)

$$[\psi] = [\psi'] = 0 \quad \text{at} \quad z = 0, L, \tag{3.26}$$

where the square bracket denotes the jump in the quantity within.

Now the solution of (2.4) and (3.9) in the uniform regions I and III are

$$\left.\begin{aligned} \psi_1 &= \exp\{\mathrm{i}k_{z1}z\} + R\exp\{-\mathrm{i}k_{z1}z\}, \\ \psi_3 &= T\exp\{\mathrm{i}k_{z3}z\} \end{aligned}\right\} \tag{3.27}$$

where k_{zi} ($i = 1, 3$) are given by

$$k_{z1}^2 = \frac{(k^2+l^2)\,(\hat{\omega}_i^2 - l^2V_i^2)\,(N^2 - \hat{\omega}_i^2 + l^2V_i^2)}{(\hat{\omega}_i^2 - l^2V_i^2)^2 - 4\Omega^2\hat{\omega}_i^2},$$

$$\hat{\omega}_i = \omega - kU_i, \tag{3.28}$$

and k_{z1} and k_{z3} must be chosen in such a way that the incident wave, amplitude unity, transports energy *towards* the layer while the reflected wave, amplitude R, and the transmitted wave, amplitude T, transport energy *away* from the layer; in the case $k_{z3}^2 < 0$, k_{z3} must be chosen so that k_{z3} is negative. The wave-invariant (2.7) can be evaluated in regions I and III to find

$$\mathscr{A}_1 = k_{z1}(1 - |R|^2), \quad \mathscr{A}_3 = k_{z3}|T|^2. \tag{3.29}$$

Since equation (2.4) is regular at $z = 0, L$ then the wave-invariant is continuous there. If the shear layer is free of critical levels, the invariance of $\mathscr{A}$ yields

$$|R|^2 = 1 - (k_{z3}/k_{z1})|T|^2. \tag{3.30}$$

This equation immediately shows that the reflected wave is amplified (i.e. $|R| > 1$) if $k_{z3}/k_{z1} < 0$. Our immediate objective is to determine whether this condition can be obeyed by any wave in the absence of a critical level within the shear layer. The wave normal surfaces will prove helpful in this respect in the sense that one can see where the prospective waves in regions I and III, which are likely to interact to give $k_{z3}/k_{z1} < 0$, are. For example, if in figure 2*a* we can have a magnetic-velocity shear in which l_g moves *away* from the axis and at the same time l_A moves *towards* the axis as the height increases (or decreases) it will be possible for some waves to be over-reflected. Indeed when the condition for this, and for similar situations in figure 2*b*, to occur is analysed, it is found that wave amplification can take place in the following situations:

(i) If the conditions

$$N^2 < \omega^2 < 4\Omega^2, \quad \frac{V_3}{V_1} < \frac{(4\Omega^2 - N^2)^{\frac{1}{2}}}{\omega},$$

$$|U_3| > \frac{V_1}{\omega}\left[\left(N^2 + \frac{\omega^2V_3^2}{V_1^2}\right)^{\frac{1}{2}} - \omega\right], \quad kU_3 < 0, \tag{3.31}$$

are satisfied, then all waves with k and l such that

$$\min\left\{\frac{[\omega(\omega+2\Omega)]^{\frac{1}{2}}}{V_1}, \frac{(\hat{\omega}_3^2-N^2)^{\frac{1}{2}}}{V_3}\right\} > |l| > \frac{\omega}{V_1}, \tag{3.32}$$

are over-reflected.

(ii) If the conditions

$$N^2 < \omega^2 < 4\Omega^2, \quad \frac{V_3}{V_1} > N(\omega^2-N^2)^{-\frac{1}{2}},$$

$$|U_3| > \omega V_1(\omega^2-N^2)^{-\frac{1}{2}}-V_3, \quad kU_3 > 0, \tag{3.33}$$

are satisfied, then all waves with k and l in the range

$$\max\left(0, \frac{|\hat{\omega}_3|}{V_3}\right) < |l| < \min\left\{\frac{(\omega^2-N^2)^{\frac{1}{2}}}{V_1}, \frac{[\hat{\omega}(\hat{\omega}+2\Omega)]^{\frac{1}{2}}}{V_3}\right\}, \tag{3.34}$$

are over-reflected.

(iii) If

$$N^2 < 4\Omega^2 < \omega^2,$$

$$\frac{N^2}{4\Omega^2}\frac{N^2}{\omega^2-N^2} < \frac{V_3^2}{V_1^2} < \frac{N^2}{\omega^2-N^2},$$

$$|U_3| > V_1-V_3(\omega^2-N^2)^{\frac{1}{2}}/\omega, \quad kU_3 > 0, \tag{3.35}$$

then all waves possessing k and l such that

$$\max\left\{\frac{|\hat{\omega}|}{V_3}, \frac{[\omega(\omega-2\Omega)]^{\frac{1}{2}}}{V_1}\right\} < |l| < \frac{(\omega^2-N^2)^{\frac{1}{2}}}{V_1}, \tag{3.36}$$

have an amplified reflected wave.

(iv) If

$$\omega^2 > \max(N^2, 4\Omega^2, N^4/4\Omega^2),$$

$$\frac{V_3^2}{V_1^2} > \max\left[\frac{4\Omega^2-N^2}{\omega^2}, \frac{N^2(N^2-4\Omega^2)}{4\Omega^2\omega^2}\right],$$

$$|U_3| > \frac{V_1}{\omega}\left(N^2+\frac{\omega^2V_3^2}{V_1^2}\right)^{\frac{1}{2}}-V_1, \quad kU_3 < 0, \tag{3.37}$$

then all waves with k and l satisfying

$$\min\left\{\frac{[\omega(\omega+2\Omega)]^{\frac{1}{2}}}{V_1}, \frac{(\hat{\omega}_3^2-N^2)^{\frac{1}{2}}}{V_3}\right\} > |l| > \frac{\omega}{V_1}, \tag{3.38}$$

are amplified.

Note that it has been assumed that $\omega > 0$ *and* V *is always positive since the transformation* $V \to -V$ *leaves the relevant equations unchanged.*

The examination of these conditions for over-reflexion shows that the transmitted wave can be either an Alfvén wave modified by rotation and gravity or an inertial wave modified by magnetic and gravitational effects. Since the study of gravity waves incident upon a finite velocity shear (Eltayeb & McKenzie 1975) shows that wave amplification is not possible in the absence of critical levels within the shear layer, it will be instructive to investigate the cause of wave amplification here.

The investigation of the wave normal surfaces (see figures 4 and 5) in the absence of a magnetic field or rotation shows that, in either case, wave amplification is not possible at least in the absence of critical levels. Thus the cause of wave amplification is the *simultaneous* action of magnetic and rotational effects.

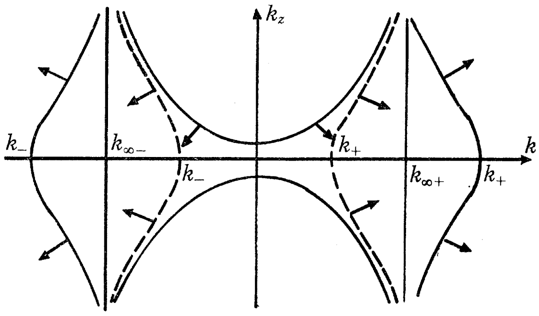

FIGURE 4. The wave normal curves for two dimensional inertial-gravity waves ($V = 0$). The continuous lines refer to $N^2 > \omega^2 > 4\Omega^2$ and the discontinuous lines apply for $N^2 < \omega^2 < 4\Omega^2$. Note that the possible ray trajectories here are of the forms (*a*) and (*c*) of figure 3. $k_\pm = (\omega \pm N)/U$, $k_{\infty\pm} = (\omega \pm 2\Omega)/U$.

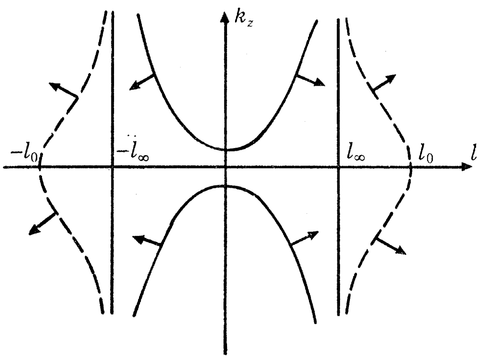

FIGURE 5. The cross section of the wave normal surfaces for hydromagnetic-gravity waves ($\Omega = 0$) in the (l, k_z) plane. The continuous (discontinuous) curves apply for $N^2 \gtrless \hat{\omega}^2$. The ray trajectories that can arise are of the form (*c*) of figure 3 for the former and of the form (*a*) for the latter. $l_0 = (\hat{\omega}^2 - N^2)^{\frac{1}{2}}/V$, $l_\infty = \hat{\omega}/V$.

When one critical level of the type $a = 0$ exists within the layer, the wave-invariant is discontinuous there and the expressions (3.20) and (3.29) yield (in the notation of § 3.3).

$$|R|^2 = 1 - \frac{k_{z3}}{k_{z1}} |T|^2 \lambda, \quad \lambda = \frac{1 - |B/A|^2}{\exp\{\pm 2\pi E\} - |B/A|^2 \exp\{\mp 2\pi E\}}. \tag{3.39}$$

Inspection of the wave normal surfaces shows that the transmitted wave will be evanescent (i.e. $k_{z3}^2 < 0$) if the wave approached the critical level from the propagating side, provided the conditions $V = 0$, $N^2 > \hat{\omega} > 4\Omega^2$ are not satisfied simultaneously (see figure 2*a*). In this situation $|R|^2 = 1$ (i.e. perfect reflexion) although the wave is still propagating on the far side of the critical level as is clear from (3.17). However, if the wave approaches the critical level from the non-propagating side, as would be the case, for example, in a wind increasing with height if $k_{A1} < 0$, the wave may emerge as a propagating wave ($k_{z3}^2 > 0$) in region III. In such cases the condition for wave amplification will depend on the sign of λ as well as on those of k_{z1} and k_{z3}.

As an illustration we shall consider a layer in which the magnetic field is uniform (but necessarily non-zero) throughout the medium and $U(>0)$ increases steadily from zero at $z = 0$ to U_3 at $z = L$. Suppose that $\omega^2 < l^2V^2$ and choose k and $l\ (>0)$ such that

$$\omega + \Omega - (\Omega^2 + l^2V^2)^{\frac{1}{2}} > kU_3 \gg \omega - lV. \tag{3.40}$$

Thus a critical level of the Alfvén mode type $a = 0$ exists just below the far end of the shear layer. The expression for B/A can be written as (see Eltayeb & McKenzie 1977)

$$\frac{B}{A} = \frac{\mathrm{i}k_{z3}W_1(L) - W_1'(L)}{W_2'(L) - \mathrm{i}k_{z3}W_2(L)}, \tag{3.41}$$

where W_1 and W_2 are the two independent solutions of (2.4), (2.5) and (3.9) defined in such a way that near the critical levels they reduce to the A and B solutions respectively. Since the critical level is just below the level $z = L$, we can approximate the solution by (3.18). Thus

$$\lambda = 4k_{z3}(L - z_c)E^{-1}\exp\{-2\pi E\} \quad (E \gg 1). \tag{3.42}$$

Now for the situation (3.40) $k_{z1} < 0$ and hence wave amplification, though weak, is present for large E. Because λ represents the relative jump in the intensity of the wave across the critical level we observe that critical level absorption and wave amplification for hydromagnetic-inertial-gravity waves can occur simultaneously; a result which is consistent with the findings of Eltayeb & McKenzie (1975) for gravity waves incident upon a shear layer.

When more than one critical level of the type $a = 0$ exists within the layer, the problem becomes more complicated for general U and V and we will not study it here.

4. Magneto-acoustic-gravity waves

4.1. *Formulation*

Consider an isothermal, inviscid, perfectly conducting medium. Choose a cartesian system of coordinates in such a way that the z-axis is vertically upwards. Then the equations of motion, continuity, induction, energy and Gauss' law allow a basic state in which the density obeys (3.3) and

$$\boldsymbol{u}_0 = U(z)\,\hat{\boldsymbol{x}}, \quad \boldsymbol{B}_0 = B(z)\,\hat{\boldsymbol{x}},$$

$$\frac{\partial p_0}{\partial z} = -\mu^{-1}BB' - \rho_0 g, \tag{4.1}$$

in the notation of the preceding section.

The equation governing the vertical component, W, of the velocity for two dimensional infinitesimal disturbances (given by (3.5) with $l = 0$) is (2.1) with

$$\begin{aligned}
a &= \hat{\omega}^2(c^2+V^2) - k^2V^2c^2,\\
b &= -\beta a + VV'(\hat{\omega}^2 - k^2c^2) + \frac{kU'(\hat{\omega}^2-k^2c^2)}{\hat{\omega}}\left[V^2 + \frac{\hat{\omega}^4c^2}{(\hat{\omega}^2-k^2c^2)^2}\right],\\
c_1 &= (\hat{\omega}^2 - k^2V^2)(\hat{\omega}^2-k^2c^2) + k^2c^2N^2 + \frac{kU''a}{\hat{\omega}} - \frac{\beta kU'a}{\hat{\omega}}\\
&\quad - \frac{2k^3gc^2\hat{\omega}U'}{(\hat{\omega}^2-k^2c^2)} + \frac{2k^2U'^2[V^2(\hat{\omega}^2-k^2c^2)^2 + c^2\hat{\omega}^4]}{\hat{\omega}^2(\hat{\omega}^2-k^2c^2)}\\
&\quad + 2kU'VV'(\hat{\omega}^2-k^2c^2)/\hat{\omega},
\end{aligned} \tag{4.2}$$

where c is the velocity of sound (assumed uniform). The x-component of $\boldsymbol{u}$, the perturbation fluid pressure p and the magnetic induction (b_x, b_z) are given by

$$u = [(kg - \hat{\omega}U')W - kc^2W']/\mathrm{i}(\hat{\omega}^2 - k^2c^2),$$

$$p = \rho_0 \frac{[g\hat{\omega}^2 - \tfrac{1}{2}\beta V^2(\hat{\omega}^2 - k^2c^2) - kc^2\hat{\omega}^2U']W - c^2\hat{\omega}^2W'}{\mathrm{i}\hat{\omega}(\hat{\omega}^2 - k^2c^2)},$$

$$b_x = -\frac{(B'\hat{\omega} + kBU')}{\mathrm{i}\hat{\omega}^2}W - \frac{BW'}{\mathrm{i}\hat{\omega}}, \quad b_z = -\frac{kB}{\hat{\omega}}W. \tag{4.3}$$

Thus in general the equation for W is singular at

$$\hat{\omega}^2 = 0,\ k^2c^2,\ k^2c^2V^2/(c^2 + V^2).$$

However, when the transformation (2.3) is used, we find that

$$\begin{aligned} C = {} & \frac{(\hat{\omega}^2 - k^2c^2)(\hat{\omega}^2 - k^2V^2) + k^2c^2N^2}{a} - \tfrac{1}{4}\beta^2 \\ & + \frac{k^3\hat{\omega}c^4U'(\beta - 2g/c^2)}{a(\hat{\omega}^2 - k^2c^2)} + \frac{\beta VV'(\hat{\omega}^2 - k^2c^2)}{a} - \frac{k^3c^4\hat{\omega}U''}{a(\hat{\omega}^2 - k^2c^2)} \\ & - \frac{VV''(\hat{\omega}^2 - k^2c^2)}{a} - \frac{2k^3c^4\hat{\omega}VV'U'}{a^2} - \frac{V'^2\hat{\omega}^2c^2(\hat{\omega}^2 - k^2c^2)}{a^2} \\ & - k^2U'^2c^4\left[\frac{k^2c^2V^2}{a^2(\hat{\omega}^2 - k^2c^2)} + \frac{3\hat{\omega}^2}{a(\hat{\omega}^2 - k^2c^2)^2}\right], \end{aligned} \tag{4.4}$$

which is not singular at $\hat{\omega} = 0$, but the other singularities of (4.2) remain. The expression for C here is again real and hence the wave-invariant is given by (2.7), (2.4) and (4.4). Written in terms of W it is

$$\mathscr{A} = \mathrm{Re}\left[\frac{-\mathrm{i}(aW'\overline{W} + b|W|^2)}{\hat{\omega}^2(\hat{\omega}^2 - k^2c^2)}\right]. \tag{4.5}$$

In analogy with gravity waves in an incompressible medium it would have been expected that the vertical transfer of horizontal momentum, taking account of the compressibility of the medium, would be proportional to $\mathscr{A}$. However, this is not the case and no simple physical interpretation of the wave-invariant is possible here.

4.2. *The wave normal curves*

When variations in the background state are neglected, a dispersion relation is obtained

$$k_z^2 = \frac{(\hat{\omega}^2 - k^2c^2)(\hat{\omega}^2 - k^2V^2) + k^2c^2N^2}{\hat{\omega}^2(c^2 + V^2) - k^2c^2V^2} - \tfrac{1}{4}\beta^2. \tag{4.6}$$

When $N = \beta = 0$, we recover the dispersion relationship for magneto-acoustic waves (see Lighthill 1960; McKenzie 1973). Now whether N and β vanish or not the wave normal curves will possess two asymptotes at

$$k = k_{\infty\pm} = \omega/[U \pm Vc(c^2 + V^2)^{-\frac{1}{2}}]. \tag{4.7}$$

Also when $N = \beta = 0$, k_z vanishes when k takes one of the values

$$k_{S1} = \frac{\omega}{U - V}, \quad k_{S2} = \frac{\omega}{U + V}, \quad k_{F1} = \frac{\omega}{U - c}, \quad k_{F2} = \frac{\omega}{U + c}. \tag{4.8}$$

Here the subscripts S and F denote slow and fast modes and the values given in (4.8) correspond to $c^2 > V^2$; if $V^2 > c^2$ then the subscripts must be interchanged (see McKenzie 1973 and figure 8). The notation of (4.8) will be adopted even if N and β do not vanish.

The most important difference between the slow and fast modes, when $\omega^2 > \frac{1}{4}\beta^2(c^2+V^2)$, is that the slow modes' wave normal curves approach asymptotes and hence their rays exhibit critical levels while the wave normal curves of fast modes do not possess asymptotes and hence their rays merely change direction at the reflexion points (see figure 11).

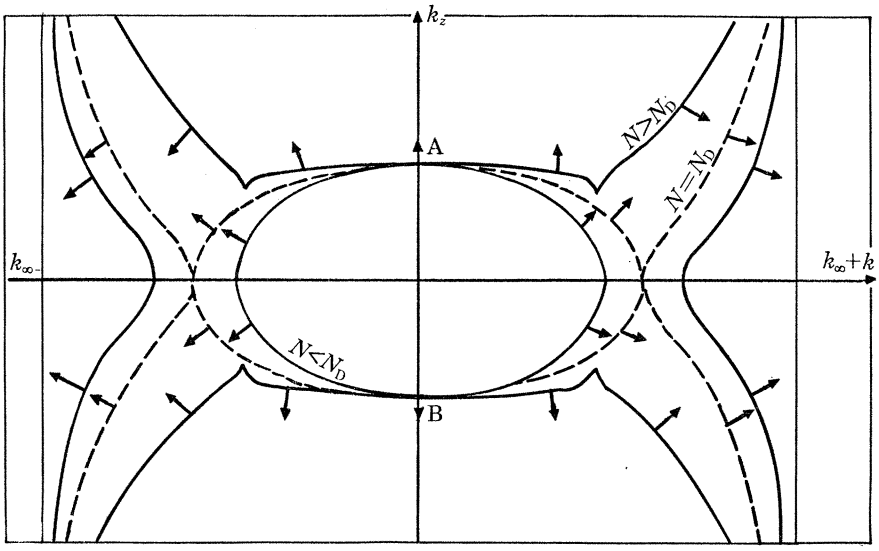

FIGURE 6. The wave normal curves for two dimensional magneto-acoustic waves (illustrated for $0 < U < Vc(c^2+V^2)^{-\frac{1}{2}}$ and $\omega > \omega_c = \frac{1}{2}\beta(c^2+V^2)^{\frac{1}{2}}$) for different values of N^2. The discontinuous curves correspond to $N^2 \equiv N_D^2$ (see equation (4.9)). Note that the two points A and B as well as the positions of the asymptotes are unaffected by N.

It is instructive, especially when discussing over-reflexion in §4.4 below, to investigate the influence of variations in U, V and N. Consider the influence of buoyancy. For fixed U, V, c and $\beta^2 < 4\omega^2/(c^2+V^2)$, the wave normal curves possess four reflexion points, two of which occur for negative k and two for positive k, when $N = 0$. As N^2 increases the two pairs of reflexion points on either side of the k_z-axis approach each other and coincide at $k = k_0$ when $N = N_D$ (we shall always assume that $N > 0$).

$$N_D^2 = \gamma^2\left[\frac{\gamma^2(c^2+V^2)}{4k_0^2c^2} - V^2\right], \quad \gamma^2 = k_0^2 + \tfrac{1}{4}\beta^2, \tag{4.9}$$

where

$$k_0 = \frac{\{2U\omega \pm [4U^2\omega^2 - (2U^2-c^2-V^2)\{2\omega^2 - \frac{1}{4}\beta^2(c^2+V^2)\}]^{\frac{1}{2}}\}^{\frac{1}{2}}}{(2U^2-c^2-V^2)}. \tag{4.10}$$

Note here that N_D is a function of U and V, for fixed c and β. For $N^2 > N_D^2$, the branches of the slow and fast modes join together (see figure 6). Since the vertical components of the group velocity for the slow and fast modes have different signs then the direction of the group velocity must change abruptly at the points where the two branches meet, i.e. the two branches meet in a cusp. Now the vertical component W_g of the group velocity computed from (4.6) is

$$W_g = \frac{\partial\omega}{\partial k_z} = \frac{k_z}{\omega}\frac{[\hat{\omega}^2(c^2+V^2) - k^2c^2V^2]^2}{c^2[\hat{\omega}^4 - k^4c^2V^2 - k^2(c^2+V^2)N^2]}. \tag{4.11}$$

A similar expression for the component of group velocity in the x-direction is also obtainable. Equation (4.11) shows that W_g changes sign at $k = \tilde{k}$, where $\tilde{k}$ is a root of

$$\hat{\omega}^4 - c^2V^2k^4 - (c^2+V^2)N^2k^2 = 0. \tag{4.12}$$

In the absence of a flow the cusps are given by

$$\tilde{k}^2 = \{-(c^2+V^2)N^2 + [(c^2+V^2)^2N^4 + 4\omega^4c^2N^2]^{\frac{1}{2}}\}/2c^2V^2, \tag{4.13}$$

which shows that

$$\tilde{k}^2 \simeq \frac{\omega^4}{(c^2+V^2)N^2}, \quad N^2 \to \infty. \tag{4.14}$$

Thus the cusps *do not* vanish completely for any finite value of N. Also note the dependence of $\tilde{k}$ on V (and c) for large N.

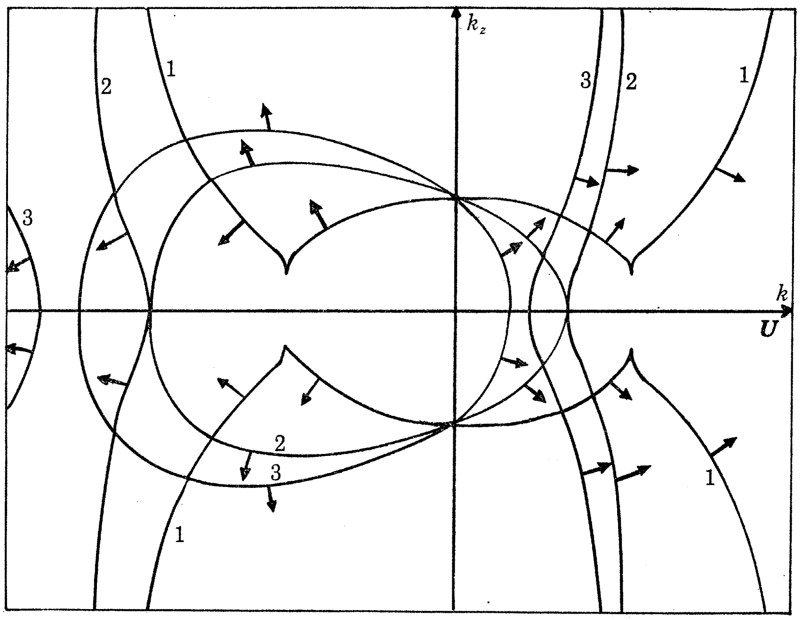

FIGURE 7. The manner in which the wave normal curves for magneto-acoustic waves in a flow, evolve with the increase of the flow (counting up). The curves start with $N^2 > N_D^2$.

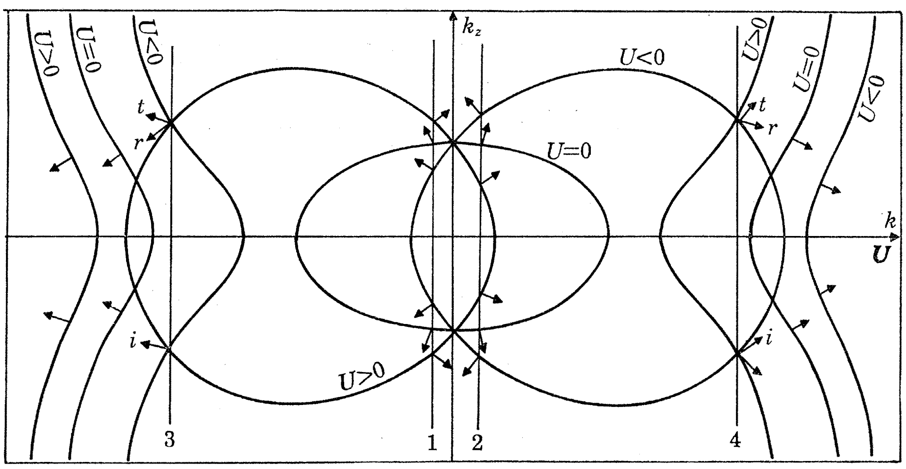

FIGURE 8. The wave normal curves for magneto-acoustic waves for different values of U for fixed V and c when $N < N_D$ and $|U| < |Vc(c^2+V^2)^{-\frac{1}{2}}|$. The ordinates 1 and 2 are used to trace the ray trajectories 1 and 2 of figure 11 while the ordinates 3 and 4 correspond to possible amplifying waves (see §4.4).

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If there is a flow the cusps will still persist for large values of N but the value of N_D increases with $|U|$. If $N > N_D$ for no flow and if the flow is increased then the cusps will move towards the k-axis and at the same time the cusp at $\tilde{k} > 0$ moves towards the k_z-axis while the one at $\tilde{k} < 0$ moves away from the k_z-axis (see figure 7). For values of $N < N_D$ or for large enough values of $|U|$, the wave normal curves evolve in the manner illustrated in figures 8 and 9.

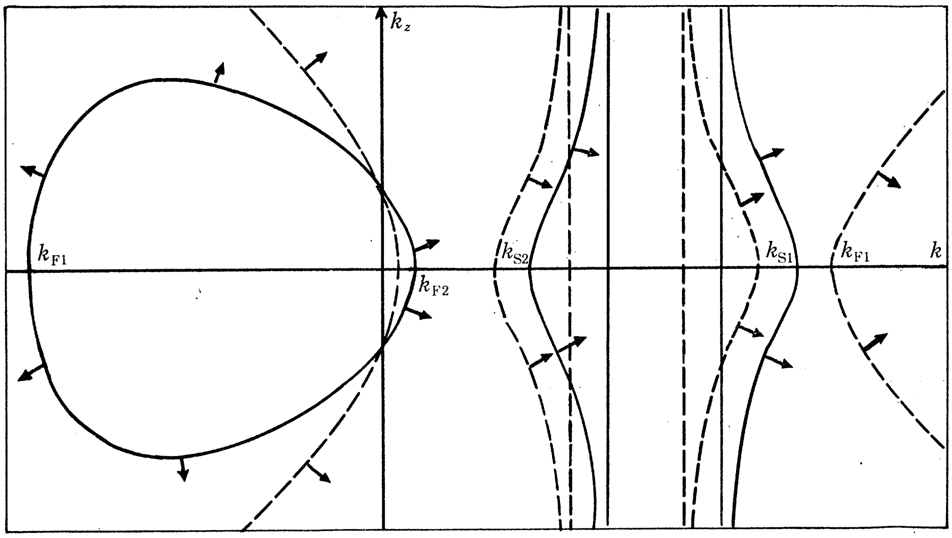

FIGURE 9. The wave normal curves for magneto-acoustic waves for $|U| > |Vc(c^2+V^2)^{-\frac{1}{2}}|$. The continuous lines correspond to min $(c^2, V^2) < U^2 <$ max (c^2, V^2) and the discontinuous lines correspond to $U^2 >$ max (c^2, V^2). The illustration is for $U > 0$. The wave normal curves for $U < 0$ are the mirror reflexions of these in the k_z-axis.

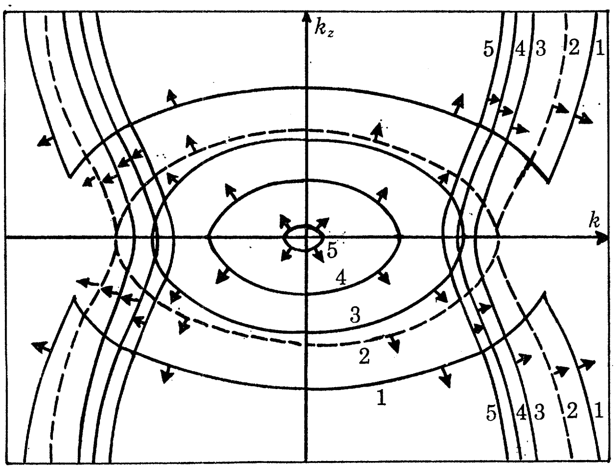

FIGURE 10. The evolution of the wave normal curves for magneto-acoustic-gravity waves (counting up) with the increase of V^2.

When V^2 is increased, keeping the other parameters fixed, the wave normal curves evolve in a slightly different manner. If $N \leqslant N_D$, the fast (acoustic) modes shrink towards the k_z-axis and eventually disappear. If, however, $N > N_D$, then as V^2 is increased the cusps move towards the k_z and k-axes and reach the k-axis at $k = k_0$ [cf equation (4.10)]. For still larger values of V^2 the slow and fast modes behave in the same manner as for $N \leqslant N_D$ described above (see figure 10).

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In contrast with hydromagnetic-inertial-gravity waves not all possible ray trajectories of hydromagnetic-acoustic-gravity waves in a velocity shear exhibit critical levels. The slow modes (which are Alfvén waves modified by gravity and compressibility) do exhibit critical levels but the fast (acoustic) modes do not (see figure 11). When $N > N_D$, a slow mode, rather than be reflected towards a critical level, can transform into a fast mode (at the cusp) and propagate upwards before it eventually propagates with the flow. On the other hand, a fast mode can transform to a slow mode and propagate towards a critical level (see figure 12). Similar behaviour also occurs in magnetic shears (see figure 13).

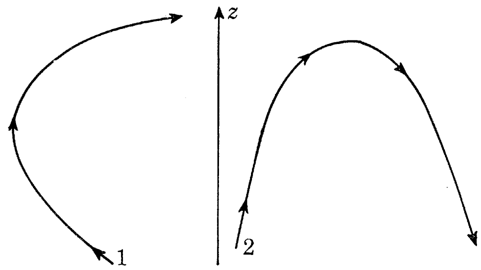

FIGURE 11. The ray trajectories arising in a magnetic-velocity ($U > 0$) shear in which both magnetic field and velocity increase with height. If the velocity and field decrease with height, the rays are obtained from these by the reflexion of 1 in the line joining its ends and by reversing the arrows in 2 (see figure 9).

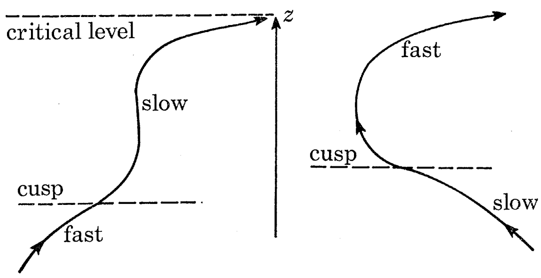

FIGURE 12. The rays for waves in a magnetic-velocity shear with k satisfying $|\tilde{k}| > |k| > |k|$ $(N > N_D)$. These waves, for each of which there corresponds a ray in the opposite direction, can give rise to wave amplification.

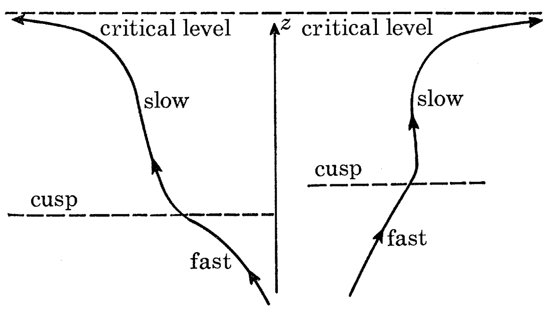

FIGURE 13. Ray trajectories arising in a magnetic shear increasing with height in an isothermal fluid. In addition to the ones shown here there are also the type 2 of figure 9 and the type (a) of figure 3. It is shown in § 4.4 that these rays can give rise to amplified reflected waves.

4.3. *Critical levels*

The wave-invariant is continuous everywhere except possibly at the singularities of (2.4) and (4.4), i.e. at $a = \hat{\varpi}^2 - k^2c^2 = 0$. It should be noted here that the singularity at $\hat{\omega}^2 = k^2c^2$ is absent not only in the W.K.B.J. treatment but also if there is no shear flow even if the magnetic shear is present.

In the neighbourhood of $a = 0$, we have

$$\psi'' + \tfrac{1}{4}\psi/(z-z_c)^2 = 0. \tag{4.15}$$

The legitimate solutions on either side of the critical level are (cf. Eltayeb & McKenzie 1977)

$$\psi = \begin{cases} (z_c - z)^{\frac{1}{2}}\{A + B\ln(z_c - z)\} & (z < z_c) \\ \mp\mathrm{i}(z-z_c)^{\frac{1}{2}}\{A + B[\ln(z-z_c) \mp \mathrm{i}\pi\}] & (z > z_c) \end{cases}, \tag{4.16}$$

where the upper (lower) sign is taken if $\kappa = \omega\hat{\omega}_c V_c(\hat{\omega}_c^3 V_c' + k^3 V_c^3 U_c') \gtrless 0$. The evaluation of the wave-invariant below and above the critical level yields

$$\mathscr{A}_{\text{below}} = \mathrm{Re}\,(-\mathrm{i}\bar{A}B),$$
$$\mathscr{A}_{\text{above}} = \mathscr{A}_{\text{below}} \pm \pi|B|^2. \tag{4.17}$$

Near the singularities $\hat{\omega}^2 = k^2c^2$, we have

$$\psi'' - \tfrac{3}{4}\psi/(z-z_c)^2 = 0, \tag{4.18}$$

which is identical to (3.22). The solution is then similar to (3.23) and thus the wave-invariant is *continuous* across this singularity. If we remember that this singularity is similar to that discussed in the preceding case when $\hat{\omega}^2 = l^2V^2$ in that both singularities are absent in the W.K.B.J. treatment, it is tempting to speculate that the wave-invariant may be continuous except at singularities predicted by the W.K.B.J. approximation but see §5 below.

4.4. *Reflexion by and transmission through a magnetic-velocity shear*

Suppose that the basic state (4.1) in the medium described at the beginning of this section is given by (3.25). The solutions in regions I and III are then given by (3.26) and (3.27) provided k_{z1} and k_{z3} satisfy

$$k_{zi}^2 = \frac{(\hat{\omega}_i^2 - k^2c^2)(\hat{\omega}_i^2 - k^2V_i^2) + k^2c^2N^2}{\hat{\omega}_i^2(c^2+V_i^2) - k^2c^2V_i^2} - \tfrac{1}{4}\beta^2 \quad (i = 1, 3) \tag{4.19}$$

and the choice of signs of k_{z1} and k_{z3} is adhered to.

Now if no critical levels exist within the layer, the wave-invariant is continuous everywhere and we get

$$|R|^2 = 1 - |T|^2(k_{z3}/k_{z1}). \tag{4.20}$$

The inspection of the wave normal curves (see figures 7–10) shows that wave amplification is possible (i.e. $k_{z3}/k_{z1} < 0$) in the absence of critical levels and that it always occurs as a result of the interaction of the slow and fast modes in regions I and III. However, before we write down the conditions for wave amplification, we need to adopt a convenient notation. In region I, $U = 0$ and we have

$$\left.\begin{aligned} k_{F1}^{(1)} &= -k_{F2}^{(1)} = -\{[f-(f^2-e)^{\frac{1}{2}}]/2c^2V_1^2\}^{\frac{1}{2}}, \\ k_{S1}^{(1)} &= -k_{S2}^{(1)} = -\{[f+(f^2-e)^{\frac{1}{2}}]/2c^2V_1^2\}^{\frac{1}{2}}, \end{aligned}\right\} \tag{4.21}$$

where the superscript refers to region I, and

$$\left.\begin{aligned} f &= \omega^2(c^2+V_1^2) - c^2N^2 - \tfrac{1}{4}\beta^2c^2V_1^2, \\ e &= 4c^2V^2\omega^2[\omega^2 - \tfrac{1}{4}\beta^2(c^2+V_1^2)], \end{aligned}\right\} \tag{4.22}$$

provided $N \leqslant N_D^{(1)}$, where $N_D^{(1)}$ is the value of N_D as given in equation (4.9), when $U = 0$ and $V = V_1$, e being positive (we shall not deal with $e \leqslant 0$ for simplicity). For values of $N > N_D^{(1)}$ the values (4.21) are complex and the fast and slow modes meet in cusps at $k = \pm\tilde{k}^{(1)}$, where $\tilde{k}^{(1)}$ is the positive root of (4.13). The asymptotes will be denoted by $k^{(1)}_{\infty\pm}$.

In region III, $k^{(3)}_{\mathrm{F1,2}}$ and $k^{(3)}_{\mathrm{S1,2}}$ are the roots of

$$(\hat{\omega}_3^2 - k^2c^2)\,(\hat{\omega}_3^2 - k^2V_3^2) + k^2c^2N^2 - \tfrac{1}{4}\beta^2[\hat{\omega}_3^2(c^2+V_3^2) - k^2c^2V_3^2] = 0,$$

$$\hat{\omega}_3 = \omega - kU_3, \tag{4.23}$$

if $N \leqslant N^{(3)}_{\mathrm{D}}$, where $N^{(3)}_{\mathrm{D}}$ is the value of N_{D}, as given in equation (4.9), when $U = U_3$ and $N = V_3$. If $N > N^{(3)}_{\mathrm{D}}$, the fast and slow modes meet in cusps at $\tilde{k}^{(3)}_{\pm}$, where these are the two real roots of (4.12) when $\hat{\omega} = \hat{\omega}_3$ and $V = V_3$. The asymptotes in region III will be denoted by $k^{(3)}_{\infty\pm}$. Also we shall refer to the slow and fast modes in both regions I and III by the subscripts of their reflexion points. For example, the branch with a reflexion point k_{S1} will be denoted 'the S1 mode'.

Now the conditions for wave amplification can be written algebraically as (see also figures 7 and 8).

(i) An incident S1 mode in a westerly wind $(U > 0)$ increasing with height will give rise to a transmitted F1 mode and an amplified reflected S1 mode provided

$$k^{(1)}_{\mathrm{S1}} > k^{(3)}_{\mathrm{F1}} \quad \text{if} \quad N \leqslant N^{(3)}_{\mathrm{D}}, \tag{4.24}$$

$$k^{(1)}_{\mathrm{S1}} > -\tilde{k}^{(1)} \quad \text{if} \quad N^{(3)}_{\mathrm{D}} < N < N^{(1)}_{\mathrm{D}}, \tag{4.25}$$

$$\tilde{k}^{(3)}_{-} > -\tilde{k}^{(1)} \quad \text{if} \quad N \geqslant N^{(1)}_{\mathrm{D}}, \tag{4.26}$$

and then the amplifying waves will have zonal wavenumbers k lying in the respective ranges

$$\max\,(k^{(3)}_{\mathrm{F1}}, k^{(1)}_{\infty-}) < k < k^{(1)}_{\mathrm{S1}} \quad \text{if} \quad N \leqslant N^{(3)}_{\mathrm{D}}, \tag{4.27}$$

$$\max\,(\tilde{k}^{(3)}_{-}, k^{(1)}_{\infty-}) < k < k^{(1)}_{\mathrm{S1}} \quad \text{if} \quad N^{(3)}_{\mathrm{D}} < N < N^{(1)}_{\mathrm{D}}, \tag{4.28}$$

$$\max\,(\tilde{k}^{(3)}_{-}, k^{(1)}_{\infty-}) < k < -\tilde{k}^{(1)} \quad \text{if} \quad N \geqslant N^{(1)}_{\mathrm{D}}. \tag{4.29}$$

(ii) An incident F2 mode in a westerly wind increasing with height and a magnetic field increasing with height will give rise to a transmitted S2 mode in region III and an amplified reflected F2 mode provided

$$k^{(3)}_{\mathrm{F2}} < k^{(1)}_{\mathrm{S2}} \quad \text{if} \quad N \leqslant N^{(3)}_{\mathrm{D}}, \tag{4.30}$$

$$k^{(3)}_{\mathrm{F2}} < \tilde{k}^{(1)} \quad \text{if} \quad N^{(3)}_{\mathrm{D}} \leqslant N < N^{(1)}_{\mathrm{D}}, \tag{4.31}$$

$$\tilde{k}^{(3)}_{+} < \tilde{k}^{(1)} \quad \text{if} \quad N \geqslant N^{(1)}_{\mathrm{D}}. \tag{4.32}$$

In each case the incident waves must have the respective zonal wavenumbers

$$k^{(3)}_{\mathrm{F2}} < k < \min\,(k^{(3)}_{\infty+}, k^{(1)}_{\mathrm{S2}}) \quad \text{if} \quad N \leqslant N^{(3)}_{\mathrm{D}}, \tag{4.33}$$

$$\tilde{k}^{(3)}_{+} < k < \min\,(k^{(3)}_{\infty+}, k^{(1)}_{\mathrm{S2}}) \quad \text{if} \quad N^{(3)}_{\mathrm{D}} < N < N^{(1)}_{\mathrm{D}}, \tag{4.34}$$

$$k^{(3)}_{+} < k < \min\,(k^{(3)}_{\infty+}, \tilde{k}^{(1)}) \quad \text{if} \quad N \geqslant N^{(1)}_{\mathrm{D}}. \tag{4.35}$$

(iii) In an easterly wind $(U < 0)$ increasing with height and a magnetic field increasing vertically, the conditions for wave amplification are obtained by imagining that regions I and III are interchanged and working out the same conditions as in the previous cases. This amounts to turning figures 7 and 8 through an angle of 180°.

One important result emerging from the conditions (4.24) to (4.35) is that when $N > N^{(1)}_{\mathrm{D}}$ *almost any non-zero* flow in region III will give rise to wave amplification.

When we compare the results of this section with those of the preceding one, we find that the rôle played by the angular velocity in making wave amplification in the absence of critical levels within the layer possible is here assumed by compressibility. It is then informative to examine the rôle of compressibility in favouring wave amplification.

If the medium is incompressible (i.e. $c^2 \to \infty$) the dispersion relation (4.6) takes the form

$$k_z^2 = \frac{k^2[N^2 - \hat{\omega}^2 + k^2V^2]}{\hat{\omega}^2 - k^2V^2} - \tfrac{1}{4}\beta^2, \tag{4.36}$$

and the inspection of the wave normal curves (see figure 14) shows that k_{z3}/k_{z1} is always positive unless a critical level exists within the layer. This shows that compressibility is an essential ingredient for wave amplification in the present circumstances.

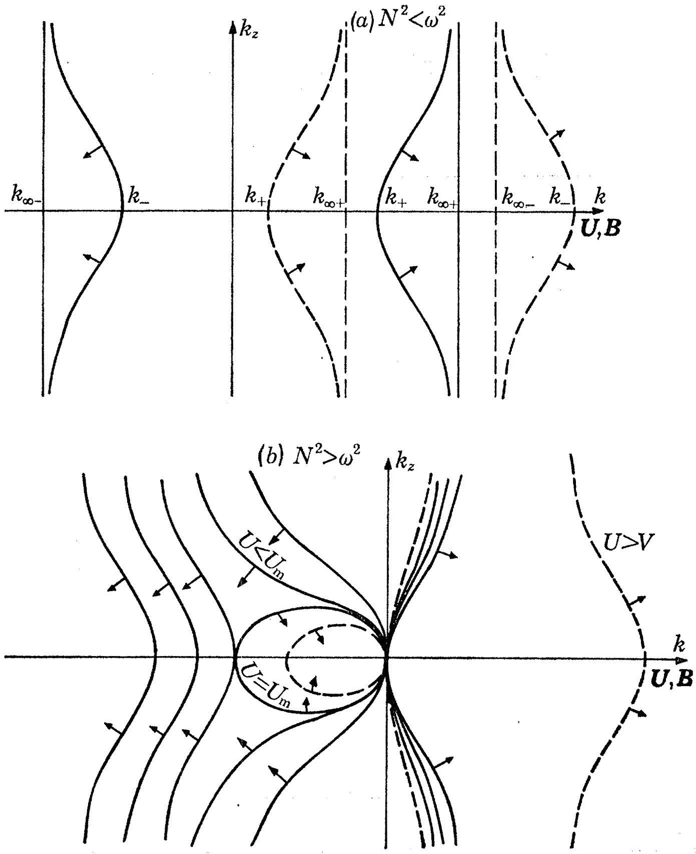

FIGURE 14. The wave normal curves for hydromagnetic-gravity waves in an incompressible Boussinesq fluid (i.e. $c^2 \to \infty$), when $\frac{1}{4}\beta^2$ is negligible, in the presence of a flow. In both (a) and (b) $k_{\infty\pm} = (\omega \pm N)/U$ and $k_\pm = \{\omega U \pm [\omega^2 U^2 - (\omega^2 - N^2)(U^2 - V^2)]^{\frac{1}{2}}\}/(U^2 - V^2)$. In (b) $U_m = (1 - \omega^2/N^2)^{\frac{1}{2}} V$. The continuous (discontinuous) curves in (b) refer to $U \lessgtr V$.

In the absence of the magnetic field, on the other hand, (i.e. $V = 0$) equation (4.6) reduces to

$$k_z^2 = \frac{\hat{\omega}^2 - k^2c^2}{c^2} + \frac{k^2N^2}{\hat{\omega}^2} - \tfrac{1}{4}\beta^2. \tag{4.37}$$

The wave normal curves are sketched in figure 15. In region I, where no flow is present, the incident and reflected waves are acoustic-gravity waves. However, in the shear and in region III, these are modified by the flow and they split into acoustic waves (slightly modified by gravity) and gravity waves (slightly modified by compressibility). Assuming $\frac{1}{4}\beta^2c^2$ is small compared with ω^2 (for $\frac{1}{4}\beta^2c^2 > \omega^2$ see McKenzie 1972), we find that $N_D^2 = \omega^2$ for both regions I and III. If $N^2 < \omega^2$ then an acoustic wave propagating with the flow and emerging as a gravity wave in region III will give rise to an amplified reflected acoustic wave in region I. The condition for over-reflection is that

$$k_2 < \frac{\omega^2}{c(\omega^2 - N^2)^{\frac{1}{2}}}, \tag{4.38}$$

where $k = k_2$ is the *intermediate* positive root of

$$(\omega - k|U|)^4 - k^2c^2(\omega - k|U|)^2 + k^2c^2N^2 = 0. \tag{4.39}$$

When (4.38) is satisfied, all waves with k such that

$$k_2 < k < \min\left(\left|\frac{\omega}{U_3}\right|, \frac{\omega^2}{c(\omega^2 - N^2)^{\frac{1}{2}}}\right), \tag{4.40}$$

will be amplified.

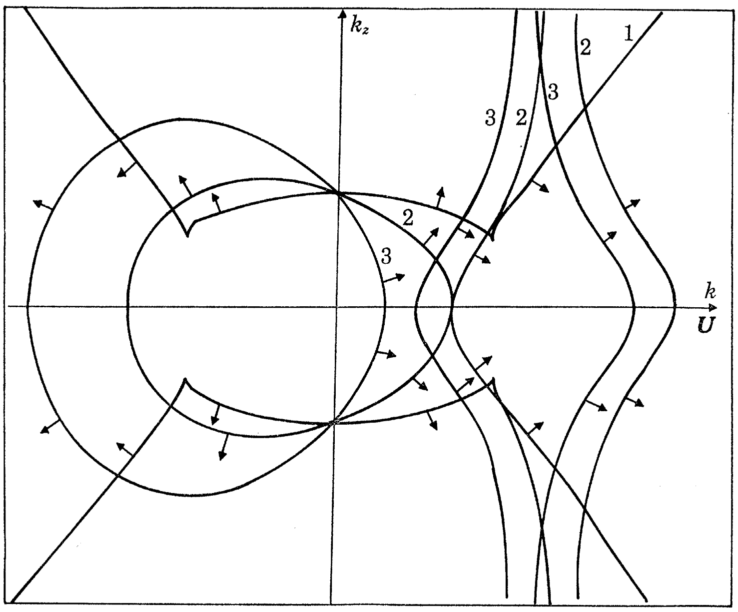

FIGURE 15. The wave normal curves for acoustic-gravity waves ($V = 0$) in the presence of a shear flow for $N^2 > \omega^2$. The curve labelled 1 corresponds to no flow and the increasing numbers refer to higher values of $U(> 0)$. Note that the cusp on the right moves towards the two axes while that on the left disappears as soon as there is a flow.

When $N^2 > \omega^2$ *wave amplification takes place for all* $|U_3| > 0$. This is because the cusp on the right of the k_z-axis moves towards the k_z-axis while the one on the left disappears completely for any $U > 0$ (if $U < 0$ the situation is reversed). The amplifying waves have zonal wavenumbers k in the ranges

$$\tilde{k}_+^{(3)} < k < \min(\tilde{k}^{(1)}, |\omega/U_3|) \quad \text{if} \quad N^2 > \omega^2, \tag{4.41}$$

$$k_2 < k < \min(\tilde{k}^{(1)}, |\omega/U_3|) \quad \text{if} \quad N^2 < \omega^2, \tag{4.42}$$

$$|k_1| < |k| < |\tilde{k}^{(1)}| \quad \text{for all } N, \tag{4.43}$$

where k_2 is the intermediate positive root of

$$(\omega - k|U_3|)^4 - k^2c^2(\omega - k|U_3|)^2 + k^2c^2N^2 = 0, \tag{4.44}$$

and k_1 is the only negative root of (4.44) if $U^2 < c^2$ and is $-\infty$ if $U^2 \geqslant c^2$.

It is commonly believed that wave amplification (or over-reflexion) is not possible in the absence of a shear flow. However, this belief stems from studies of the incompressible fluid. Since the medium here is compressible it is instructive to examine the conditions for wave amplification in the absence of a shear flow to see if it is possible at all that they can be satisfied. When we set $U = 0$ in (4.6), we get

$$k_z^2 = \frac{(\omega^2 - k^2c^2)(\omega^2 - k^2V^2) + k^2c^2N^2}{\omega^2(c^2+V^2) - k^2c^2N^2}, \tag{4.45}$$

where we have neglected the $\frac{1}{4}\beta^2$ term. Also assume that $U(z)$ and U_3 in (3.25) vanish.

From the wave normal curves at various heights (see figure 10) it is clear that the condition for wave amplification in a magnetic shear increasing with height is

$$|k_{\mathrm{F1}}^{(1)}| > |k_{\mathrm{S}}^{(3)}| \quad \text{for} \quad N \leqslant N_{\mathrm{D}}^{(3)}, \tag{4.46}$$

$$|\tilde{k}^{(1)} > |k_{\mathrm{S}}^{(3)}| \quad \text{for} \quad N_{\mathrm{D}}^{(3)} < N < N_{\mathrm{D}}^{(1)}, \tag{4.47}$$

$$|\tilde{k}^{(1)}| > |\tilde{k}_{\pm}^{(3)}| \quad \text{for} \quad N \geqslant N_{\mathrm{D}}^{(1)}, \tag{4.48}$$

where F and S refer to fast and slow; the subscripts 1 and 2 are dropped because of the symmetry about the k_z-axis. To show that these conditions can be met, we immediately see that when $N > N_{\mathrm{D}}^{(3)}$, and because the cusps move towards the k_z-axis, conditions (4.47) and (4.48) will be satisfied by only a small change in V. If $N \leqslant N_{\mathrm{D}}^{(3)}$, it is not obvious that condition (4.46) can be satisfied. However, if we consider the extreme case $V_3 \to \infty$, we find that

$$\left.\begin{aligned} k_{\infty\pm}^{(3)} &= \pm\frac{\omega}{V_3}\left[1 + \frac{c^2}{2V_3^2} + O\left(\frac{c^4}{V_3^4}\right)\right], \\ k_{\mathrm{S1,2}}^{(3)} &= \pm\frac{\omega}{V_3}\left[1 - \frac{c^2N^2}{\omega^2V_3^2} + O\left(\frac{c^4}{V^4}\right)\right]. \end{aligned}\right\} \tag{4.49}$$

If we realize that

$$|k_{\mathrm{S1,2}}^{(1)}| > \min\left(\left|\frac{\omega}{V_1}\right|, \left|\frac{\omega}{c}\right|\right), \tag{4.50}$$

as can be deduced from (4.45), then condition (4.46) can be obeyed at least by large values of V_3^2/V_1^2. For the range of k for which the waves are over-reflected we have

$$|k_{\mathrm{S}}^{(3)}| < |k| < \min\left(|k_{\infty\pm}^{(3)}|, |k_{\mathrm{F}}^{(1)}|\right) \quad (N \leqslant N_{\mathrm{D}}^{(3)}), \tag{4.51}$$

$$|\tilde{k}_{\pm}^{(3)}| < |k| < \min\left(\tilde{k}^{(1)}, |k_{\mathrm{F1,2}}^{(1)}|\right) \quad (N^{(3)} < N < N_{\mathrm{D}}^{(1)}), \tag{4.52}$$

$$|\tilde{k}_{\pm}^{(3)}| < |k| < \min\left(\tilde{k}^{(1)}, |k_{\infty\pm}^{(3)}|\right) \quad (N \geqslant N_{\mathrm{D}}^{(1)}). \tag{4.53}$$

These results permit us to conclude that *wave amplification can be exhibited by a steadily increasing magnetic shear, within which no critical levels exist, in a compressible isothermal medium in the absence of a shear flow.*

When a single critical level exists within the layer then from (4.17), using the invariance of $\mathscr{A}$ below and above the critical level, we get

$$|R|^2 = 1 - (k_{z3}/k_{z1})|T|^2 \pm \pi|B|^2/k_{z1} \quad \text{for} \quad \kappa \gtrless 0. \tag{4.54}$$

Thus critical levels with $\kappa > 0$ tend to favour over-reflexion if $k_{z1} > 0$ and they tend to oppose it if $k_{z1} < 0$ while critical levels with $\kappa < 0$ have the opposite effect.

If a second critical level is also present and if the solutions near it have amplitudes $\tilde{A}$ and $\tilde{B}$ it can be shown that

$$|R|^2 = 1-(k_{z3}/k_{z1})|T|^2 \pm \pi|B|^2 k_{z1} \pm \pi|\tilde{B}|^2/k_{z1} \quad (\kappa \gtrless 0, \tilde{\kappa} \gtrless 0). \tag{4.55}$$

where $\tilde{\kappa}$ is the value of κ at the second critical level. Thus if κ and $\tilde{\kappa}$ have similar (different) signs then the two critical levels supplement (oppose) each other in favouring or hindering wave amplification.

5. Hydromagnetic-gravity waves in a sheared magnetic field

5.1. *Formulation*

Consider an inviscid, infinitely conducting Boussinesq fluid. The equations of motion, continuity, induction and Gauss' law admit a basic state in which

$$\boldsymbol{u}_0 = 0, \quad \boldsymbol{B}_0 = (B_x(z), B_y(z), 0),$$
$$\nabla(p_0 + \boldsymbol{B}_0^2/2\mu) = \rho_0 \boldsymbol{g}, \tag{5.1}$$

in the usual notation, and the density ρ_0 has the property that the Brunt-Väisällä frequency

$$\boldsymbol{N} = \frac{\boldsymbol{g}}{|\boldsymbol{g}|}\left[\boldsymbol{g}.\frac{\nabla\rho_0}{\rho_0}\right]^{\frac{1}{2}} = (N_x, N_y, N_z), \tag{5.2}$$

is uniform. Note that both $\boldsymbol{B}_0$ and $\boldsymbol{g}$ are neither parallel nor perpendicular to the direction of shear of the magnetic field. This anisotropy of the system has been shown by Acheson (1973) to give rise to the valve effect. Here we shall exploit this model, because of its anisotropy, to clarify the nature of the singularities which appear solely because of the inclusion of variations in the basic state and also to examine the valve effect using a full wave treatment.

Assuming disturbances of the form (3.5), we find that the equation for the component of velocity in the z-direction, W, satisfies (2.1) provided

$$a = \omega^2 - Q^2 - N_x^2 - N_y^2,$$
$$b = -\mathrm{i}N_z\sigma - \frac{QQ'}{\alpha(\omega^2-Q^2)}\{\lambda + \sigma^2\chi^2/\lambda\},$$
$$c_1 = \chi^2 - \alpha(\omega^2 - Q^2 - N_z^2) - \frac{2\mathrm{i}N_z\sigma\chi^2 QQ'}{\lambda(\omega^2-Q^2)}, \tag{5.3}$$

where

$$\left.\begin{aligned} Q &= kV_x + lV_y, \quad (V_x, V_y) = (B_x/(\mu\rho_0)^{\frac{1}{2}}, B_y/(\mu\rho_0)^{\frac{1}{2}}),\\ \alpha &= k^2+l^2, \quad \sigma = kN_x + lN_y,\\ \chi &= lN_x - kN_y, \quad \lambda = \alpha(\omega^2-Q^2)-\chi^2.\end{aligned}\right\} \tag{5.4}$$

These equations reduce to those of Acheson (1973) when $Q' = 0$. However, the inclusion of Q' introduces two more singularities at $\lambda = \omega^2 - Q^2 = 0$. The application of the transformation (2.3) to (5.3) leads to (2.4) with

$$C = \frac{\chi^2 - \alpha(\omega^2-Q^2-N_z^2)}{a} - \frac{QQ''(\lambda^2+\sigma^2\chi^2)}{\alpha(\omega^2-Q^2)\,a} + \frac{N_z^2\sigma^2}{a^2} + \frac{Q^2Q'^2(\lambda^2+\sigma^2\chi^2)}{\alpha(\omega^2-Q^2)\lambda a^2}$$
$$- \frac{Q'^2(\omega^2+Q^2)\,(\lambda^2+\sigma^2\chi^2)}{\alpha(\omega^2-Q^2)^2\,\lambda a} + \frac{Q^2Q'^2\chi^2(\lambda^2+\sigma^2\chi^2)}{\alpha(\omega^2-Q^2)^2\lambda^2 a} + \frac{2Q^2Q'^2(\lambda^2-\sigma^2\chi^2)}{(\omega^2-Q^2)\,a\lambda^2}. \tag{5.5}$$

It is worth noting that C is real despite the anisotropy of the system. However, all the singularities of (5.3) are retained in this case.

5.2. *The wave normal surfaces*

When we set $Q' = 0$ and assume that $\psi \propto \exp\{ik_z z\}$, we find the dispersion relation

$$k_z^2 = \frac{\chi^2 - \alpha(\omega^2 - Q^2 - N_z^2)}{a} + \frac{N_z^2 \sigma^2}{a^2}. \tag{5.6}$$

This equation immediately shows that k_z is real on *both* sides of the critical level $a = 0$ unless σ or N_z vanishes (see figure 16), i.e. the waves propagate on both sides of the critical level. To consolidate this result with the findings of Acheson (1973), namely that the critical level $a = 0$ exhibits valve effects, we must recognize that the transformation (2.3) removes a propagation part from the solution. If $W \propto \exp(imz)$ then the dispersion relation in m is

$$am^2 - 2N_z\sigma m - \chi^2 + \alpha(\omega^2 - Q^2 - N_z^2) = 0, \tag{5.7}$$

(cf. equation (2.2) in Acheson (1973) but note that $(k, l) \to (-k, -l)$). The wave normal surfaces (see figure 17) in terms of m clearly show the valve effect while those in terms of k_z do not exhibit valve effects but indicate that waves approaching the critical level from either side are absorbed there. This is wholly due to the fact that the transformation (2.3), which in the W.K.B.J. approximation yields $k_z = m - \sigma N_z/a$ destroys the anistropy of the system thereby annihilating the valve effects. An examination of internal gravity waves in a shear flow when gravity makes an acute (or obtuse) angle with the direction of flow and shear (Acheson 1973, p. 32) gives similar results.

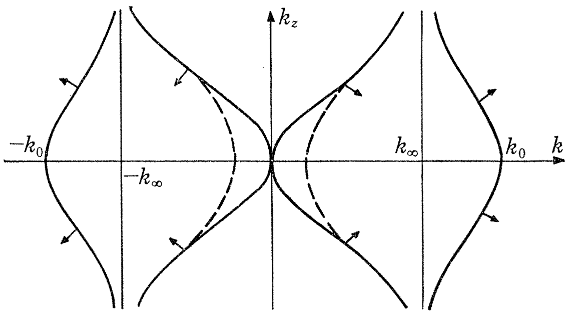

FIGURE 16. The cross-section in the (k, k_z) plane of the wave normal surfaces for hydromagnetic-gravity waves in an anisotropic medium when $l = 0$. For non-zero l slight modifications are required (see equation (5.6)). The two curves meet at the origin if $\omega^2 - Q^2 - N_y^2 > 0$ otherwise they move apart as indicated by the discontinuous lines. $k_\infty = (\omega^2 - N_x^2 - N_y^2)^{\frac{1}{2}}/V_x$, $k_0 = (\omega^2 - N_y^2)^{\frac{1}{2}}/V_x$.

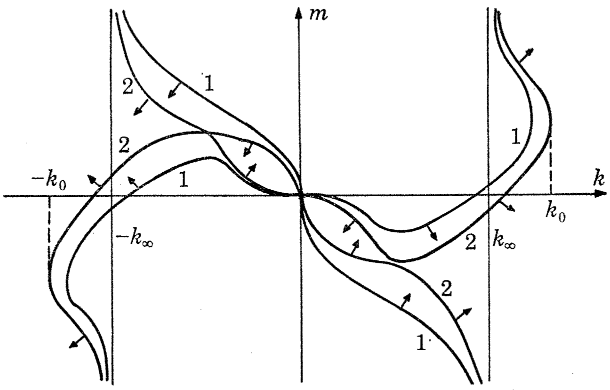

FIGURE 17. The cross section of the wave normal surfaces in the (k, m) plane (see equation (5.7)) for $l = 0$. The curves labelled 1, which do not meet except at the origin, correspond to $N_x^2 + N_y^2 + N_z^2 > \omega^2 > \max(N_x^2 + N_y^2, N_x^2 + N_z^2)$ while 2 correspond to $\omega^2 > N_x^2 + N_y^2 + N_z^2$. k_0 and k_∞ are the same as in figure 16.

5.3. *Critical levels*

Near the critical levels $a = 0$, we have

$$\psi'' + R_E \psi/(z-z_c)^2 = 0,$$
$$R_E = \tfrac{1}{4} + \mu^2, \quad \mu = N_z \sigma / 2Q_c Q_c'. \tag{5.8}$$

Here R_E is an 'equivalent Richardson number' in analogy with gravity waves in a shear flow, which is a measure of variations in the shear flow. Note that $R_E \geqslant \frac{1}{4}$ and hence the problem is in a sense similar to that of gravity waves in a shear flow when the Richardson number is greater than $\frac{1}{4}$. The solution near this critical level then is

$$\psi = \begin{cases} A(z_c - z)^{\frac{1}{2}+i\mu} + B(z_c - z)^{\frac{1}{2}-i\mu} & (z < z_c), \\ A^*(z - z_c)^{\frac{1}{2}+i\mu} + B^*(z - z_c)^{\frac{1}{2}-i\mu} & (z > z_c), \end{cases} \tag{5.9}$$

where $$A^* = \mp iA \exp(\pm \pi\mu), \quad B^* = \mp iB \exp(\mp \pi\mu), \tag{5.10}$$

in which the upper (lower) sign is taken if $\omega Q_c Q_c' \lessgtr 0$.

Although the amplitudes of the two solutions are drastically changed across the critical level, especially for large values of $|\mu|$ and despite the analogy with gravity waves in a shear flow, care must be exercised when deductions about the propagation of the waves across the level are made since the transformation (2.3) in this model involves a propagating part. Now from (2.3), (5.3) and (5.9) it can be shown that

$$W = \begin{cases} A(z_c - z)^{1+2i\mu} + B(z_c - z) & (z < z_c) \\ A_1^*(z - z_c)^{1+2i\mu} + B_1^*(z - z_c) & (z > z_c) \end{cases}, \tag{5.11}$$

in which $$A_1^* = -A \exp(\pm 2\pi\mu\}, \quad B_1^* = -B. \tag{5.12}$$

The *local* wavenumbers then are $$k_z = \frac{\mu}{z - z_c}, \quad m = \frac{2\mu}{z - z_c}, \tag{5.13}$$

for the A solution, and $$k_z = \frac{-\mu}{z - z_c}, \quad m = 0, \tag{5.14}$$

for the B solution. Thus in terms of k_z the two waves represent ascending and descending waves but in terms of m one wave is *completely absorbed* at the critical level while the other propagates right through. Thus *the valve effect is present whatever the scale of variations of the magnetic field in relation to the wavelength of the waves.*

The wave-invariant below and above the critical level is given by equation (3.20) of §3 if we replace E by μ.

In the neighbourhood of the critical levels $\lambda = 0$ and $\omega^2 - Q^2 = 0$, ψ is governed by equation (3.20) and the solution is therefore similar to (3.23). The wave-invariant is then *continuous* across both levels although the solutions are singular across the same levels.

5.4. *Reflexion by and transmission through a magnetic shear*

In the absence of critical levels of the type $a = 0$ the invariance of $\mathscr{A}$ yields equation (3.30) for the reflexion coefficient. However, as is clear from the wave normal surfaces of figures 16 and 17, it is not possible for k_{z3}/k_{z1} to be negative. In the presence of critical levels, on the other hand, it can be shown in an exactly similar way to that adopted for gravity waves when the Richardson number exceeded $\frac{1}{4}$ (see Eltayeb & McKenzie 1975), that wave amplification is impossible.

6. Concluding remarks

It has been shown here that *all* linear wave motions in one dimensional magnetic and/or velocity shears, which are governed by a second order linear ordinary differential equation, possess a quantity $\mathscr{A}$ which is independent of the coordinate in which the magnetic field and velocity vary; the quantity $\mathscr{A}$ is continuous everywhere except possibly at the points where the governing equation is singular. The search for simple physical interpretations of the wave-invariant $\mathscr{A}$ in different situations has not always been successful. (A similar difficulty was encountered by Grimshaw (1975) in his study of the critical layer absorption in a rotating fluid.) Perhaps it should not always be expected that the wave-invariant will have a simple physical meaning. The significance of $\mathscr{A}$ lies in the fact that simple physical quantities like momentum transfer, angular momentum flux and energy flux can always be expressed in terms of it.

When the wave-invariant is used as a measure of the intensity of the wave and the solutions near the singularities in various magnetic and/or velocity shears (only three of which are included above) are studied, it is found that the wave-invariant is always *discontinuous* across singularities predicted by the W.K.B.J. approximation, and is always *continuous* across singularities whose presence is solely due to the inclusion of variations in the basic state although the solutions are necessarily singular there.

The investigations of the solutions in the neighbourhoods of the critical levels giving rise to valve effects (see §5) show that near these levels the motions are usually governed by an 'equivalent Richardson number' R_E (say), i.e. the scale of the motions in the vicinity of these levels depends on a dimensionless parameter which is a measure of the scale of variations of the basic state. The valve effect is found to persist for *all* values of R_E. This indicates that the valve effect does not occur as a result of the W.K.B.J. approximation.

The studies of the reflectivity and transmitivity of different waves by *finite* magnetic and/or velocity shears show that *the reflected wave can be amplified in the absence of critical levels within the shear layer*. If the fluid is incompressible, this can only occur under the simultaneous action of a magnetic field and rotation in the presence of a shear. Wave amplification (or over-reflexion) here relies heavily on the presence of slow hydromagnetic inertial waves modified by gravitational effects (these are termed MAC-waves by Braginsky 1967) which propagate across magnetic field lines and they exist only in the presence of rotation (Hide 1966). In a compressible isothermal medium, on the other hand, wave amplification can occur in the absence of critical levels if both gravity and a magnetic shear are present, *even if a shear flow is excluded*. Here wave amplification involves the interaction of slow and fast hydromagnetic-acoustic-gravity waves (as opposed to positive and negative energy waves).

In a velocity shear it is generally accepted that the over-reflected wave extracts energy from the streaming motion. Now magnetic systems are known to possess instabilities which are wholly due to the spatial variations in the magnetic field (Acheson 1973*a*). In a magnetic shear in a compressible fluid, in the absence of a flow, it seems plausible to suggest that the over-reflected wave extracts its energy from the magnetic energy of the system. Whether this can be achieved in a magnetically stable system or not can only be decided by a stability analysis. Even if over-reflexion can occur in a stable system, a nonlinear study is necessary to clarify the manner in which the energy is extracted by the wave.

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